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EXTREMAL PROBLEMS FOR FUNCTIONS OF BOUNDED BOUNDARY ROTATION[1]

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Communicated by Alberto P. Calderón, April 19, 1967

1. **Preliminaries.** Let V_k denote the class of analytic functions in $D=\{z: |z|<1\}$ which have there the representation

$$f(z) = \int_0^z \exp\left(-\int_0^{2\pi} \log(1-\zeta e^{-i\theta})d\psi(\theta)\right)d\zeta \tag{1}$$

where $\psi(\theta)$ is a real valued function of bounded variation for $0\leqq\theta<2\pi$, satisfying there the conditions

$$\int_0^{2\pi} d\psi(\theta) = 2, \qquad \int_0^{2\pi} |d\psi(\theta)| \leqq k.$$

V_k is the class of analytic functions in D which have boundary rotation bounded by $k\pi$. Thus, V_k consists of those functions $f(z)=z+a_2z^2+\cdots$ which are analytic and satisfy $f'(z)\neq 0$ in D, and map D onto a domain having boundary rotation bounded by $k\pi$.

Briefly, the boundary rotation of a schlicht domain G with continuously differentiable boundary curve is the total variation of the direc-

[1] Research partially sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Grant No. AF-AFOSR 1077-66.

[2] The author wishes to express his gratitude to Professor Harry E. Rauch for his many helpful conversations during this research.

tion angle of the boundary tangent under a complete circuit. If the boundary of G does not satisfy sufficient smoothness conditions, the boundary rotation is defined by a limiting process. For nonschlicht domains without interior branch points, the boundary rotation is defined in a similar manner.

In the representation (1), the quantity $\pi\int_0^{2\pi}|d\Psi(\theta)|$ is the boundary rotation of $f(z)$.

Let S denote the class of functions $f(z)=z+a_2z^2+\cdots$ which are univalent in D, and let S_k denote the subclass of S consisting of those functions having boundary rotation bounded by $k\pi$. For $2\leqq k\leqq 4$, we have $S_k\equiv V_k$. For $k>4$ S_k is a subclass of V_k.

The class V_k has been studied by O. Lehto [3], and the reader is referred to that paper where references for the above remarks can be found. The class S_k has recently been studied by M. Schiffer and O. Tammi [6]. For $2\leqq k\leqq 4$, we shall be able to compare their results with ours.

We have developed a variational method for the class V_k. This method is based upon a general method due to G. M. Goluzin [1], and consists of appropriately varying the function $\Psi(\theta)$. In this note, we announce the solutions to certain general extremal problems for V_k. These solutions are obtained through applications of this method. The solution of extremal problems for V_k is equivalent to finding the $\Psi(\theta)$ which corresponds to the extremal function. This variational method as well as the details of its applications will be published elsewhere [5].

2. A general extremal problem for V_k.

THEOREM 1. *Let $F(w)$ be an entire function and let $\zeta\in D$ be a given point. The functional*

$$L(f') = \operatorname{Re} F[\log f'(\zeta)]$$

attains its maximum in V_k only for a function of the form

$$f'(z) = (1-ze^{-i\alpha})^A/(1-ze^{-i\beta})^B \tag{2}$$

where

$$0\leqq\alpha<2\pi,\qquad 0\leqq\beta<2\pi,\qquad A\leqq k/2-1,\qquad B\leqq k/2+1.$$

This solution was obtained by showing that $\psi(\theta)$ was a step function having a jump of height A at $\theta=\alpha$ and a jump of height $-B$ at $\theta=\beta$.

In the case where $F(w)$ satisfies the additional condition

$$F(au+bv) = a\,F(u)+bF(v)$$

the form of the extremal function (2) can be improved to

$$f'(z) = (1 - ze^{-i\alpha})^{(k/2)-1}/(1 - ze^{-i\beta})^{(k/2)+1}. \tag{3}$$

Setting $F(w) = \pm iw$, we obtain the "rotation theorem" for V_k.

COROLLARY 1. *The functional* $|\arg f'(\zeta)|$ *attains its maximum in* V_k *only for a function of the form* (3). *Hence, for all* $f \in V_k$ *and all* $z \in D$,

$$|\arg f'(z)| \leqq k \sin^{-1} |z| \tag{4}$$

where $\sin^{-1} 0 = 0$.

The bound (4) is the same one found by Schiffer and Tammi for S_k. By appropriate choice of $F(w)$, we also arrive at the distortion theorems of Lehto [3].

3. **Coefficient problems for** V_k. The problem of maximizing $|a_2|$ and $|a_3|$ over V_k has been solved by Lehto [3]. The extremal function was found to be (3) with $\alpha = \pi$ and $\beta = 0$. Lehto conjunctures that this function is extremal for the problem of maximizing $|a_n|$ for any n.

We consider a more general problem and obtain the following result.

THEOREM 2. *Let* $F(z_2, \cdots, z_n)$ *be any function having continuous partial derivatives in each of the variables* $z_2, \cdots, z_n$. *To each function* $f(z) = z + a_2 z^2 + \cdots \in V_k$ *associate the number*

$$C(f) = \operatorname{Re} F(a_2, \cdots, a_n).$$

Any function $F(z) \in V_k$ *which maximizes* $C(f)$ *over* V_k *must be of the form*

$$f'(z) = \prod_{j=1}^{M} (1 - z \exp(-i\theta_j))^{\alpha_j} \Big/ \prod_{j=1}^{N} (1 - z \exp(-i\phi_j))^{\beta_j} \tag{5}$$

where $M \leqq n-1$, $N \leqq n-1$,

$$\sum_{j=1}^{M} \alpha_j \leqq k/2 - 1, \qquad \sum_{j=1}^{N} \beta_j \leqq k/2 + 1,$$

$\theta_j, \phi_j \in [0, 2\pi)$ *all* j, *and* $\alpha_j \beta_j, \geqq 0$ *all* j.

We have been unable to improve this result even for the problem of maximizing $|a_n|$. However, the conjectured extremal function is of the form (5) and investigations confined to functions of the form (5) may be more fruitful. It should be pointed out that the result of Schiffer and Tammi for the problem of maximizing $|a_n|$ over S_k are similar to those stated here.

4. A remark on [4]. In the recent note [4], theorems similar to those presented here are discussed for several classes of univalent functions in D. There (Theorems 2 and 3) the additional hypothesis that $F'\ [\log f'(\zeta)]\neq 0$ is made. We should like to point out that due to a recent result of W. Kirwan [2], this is no longer necessary.

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