EMBEDDING PROJECTIVE SPACES[1]

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1. Haefliger reduced the question of embedding manifolds in the Euclidian space R^m to a homotopy problem in [6]. Since then it has been of some interest to find examples of n-manifolds which embed in R^{2n-k} for a given k. In particular great effort has been spent studying embeddings of the various projective spaces. However, the k that were thus obtained were in no cases larger than 5 or 6 (see for example [7], [8], [9]). Our purpose in this note is to indicate the proofs of the theorems that follow.

THEOREM 1. *Let* $n \equiv 7(8)$; *then* RP^n (real n-dimensional projective space) *embeds in* R^{2n-k} *where* $k \geqq 2\ [\log_2(\alpha(n))]-1$. (Here $\alpha(n)$ is the number of ones in the dyadic expansion of n.)

THEOREM 2. *If n is odd and $\alpha(n)$ is greater than* $4+2^i$, *then* CP^n (complex projective space) *embeds in* R^{4n-k} *with* $k \geqq 3+i$.

THEOREM 3. *If* $\alpha(n) \geqq 11+2^i$ *then* QP^n (quaternionic projective space) *embeds in* R^{8n-k} *where* $k \geqq 5+i$.

The detailed proof of Theorem 1 appears in [5] so in the sequel we will concentrate on giving those modifications which must be made in [5] so as to prove Theorems 2 and 3.

2. **A key lemma.** Let M^n immerse in R^{2n-r} and set $k(n)=8s+2^t-1$ (where $n+1=(2^{4s+t})c$ with c odd and $0 \leqq t \leqq 3$). Then for $n \geqq 3$ we have:

LEMMA 2.1. (a) *If n is odd there are exactly two isotopy classes of immersions $M^n \subseteq R^{2n}$. One contains an embedding and the other an immersion with a single double point as its only singularity, but both normal bundles have k independent cross-sections where* $k=\min(r, k(n))$.

(b) *If n is even and M^n orientable then there are Z isotopy classes of immersions $M^n \subseteq R^{2n}$ only one of which contains an embedding. The only immersion with a normal field is the embedding, hence the embedding has r normal fields.*

REMARK. Part b is false for nonorientable manifolds for all n [4].

PROOF. Part a follows from Whitney's well known results [10] on embeddings and immersions in R^{2n}, and a careful study of how one

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changes the number of double points in an immersion. The details are in [5]. To prove part b note that, according to Hirsch [1], the isotopy classes of immersions are in 1-1 correspondence with the homotopy classes of cross-sections of the stable ($n+1$-dimensional) normal bundle. If n is even the homotopy classes of n-plane bundles stably equivalent to the stable normal bundle are classified by $\chi/2$ (where χ is the Euler class of the bundle). But in this case χ is the obstruction to finding a cross-section. Finally, we note that the normal bundle to an embedding $M^n \subset R^{2n}$ must have Euler class equal to zero [3].

For f: $M^n \subseteq R^{n+k}$ we denote by η_f the normal bundle associated to f.

COROLLARY 2.2. *If n is even, M^n compact and orientable, and if ν is a subbundle of η_f for some immersion f: $M^n \subseteq R^{2n-k}$ with $k>0$, then η_g where g is the embedding g: $M^n \subset R^{2n}$ also contains ν as a subbundle.*

3. **Embedding bundles over projective spaces.** Using Corollary 2.2 and the immersion results of [2] we can prove:

THEOREM 3.1. (a) *If $2p<\alpha(n)-\alpha(p+1)-3$, then $\eta_{\mathrm{CP}^q \subset \mathrm{CP}^n}$ embeds in R^{4q} where $n=p+q+1$,*

(b) *If $4p<\alpha(n)-\alpha(p+1)-10$, then $\eta_{\mathrm{QP}^q \subset \mathrm{QP}^n}$ embeds in R^{8q} where $n=p+q+1$.*

The proof follows closely the arguments of §3 of [5], and in particular the argument following the proof of Lemma 3.2.

THEOREM 3.2. (a) *If n is odd then* $\mathrm{CP}^n \subset R^{4n}$ *with $\alpha(n)$ trivial sections.*

(b) *If $\alpha(n)>3$, then* $\mathrm{QP}^n \subset R^{8n}$ *with $\alpha(n)-3$ sections.*

This follows directly from the immersion results of [2] together with 2.2.

4. **Double mapping cylinders and the main theorems.** Suppose we have spaces X, Y, and Z and maps

$$f\colon Y \to Z, \qquad g\colon Y \to X$$

then the double mapping cylinder $M(f, g)$ is obtained from the disjoint union $X \cup I \times Y \cup Z$ by identifying a point $(0, y)$ in $I \times Y$ with $f(y)$ in Z and $(1, y)$ with $g(y)$ in X. The usual mapping cylinder is obtained by setting $X=Y$ and $g=\mathrm{id}$. We denote it by $M(f)$.

Let FP^n represent either CP^n or QP^n. Let FP^q be embedded in FP^n as the set of points whose last $p+1$ homogeneous coordinates are zero (where $n=p+q+1$). Embed FP^p in FP^n as the set of points whose first $q+1$ coordinates equal zero. Finally, set $E_{p,q}$ equal to the set of points with (normalized) homogeneous coordinates $\langle x_1, \cdots,$

$x_{q+1}, y_1, \cdots, y_{p+1}\rangle$ where $\sum_i x_i\bar{x}_i = \sum_j y_j\bar{y}_j = 1/2$. There are evident projections $\pi_1: E_{p,q} \to FP^p$, $\pi_2: E_{p,q} \to FP^q$, and we have

LEMMA 4.1. (a) $M(\pi_1) = \eta_{\mathrm{FP}^p \subset \mathrm{FP}^n}$,
(b) $N(\pi_2) = \eta_{\mathrm{FP}^q \subset \mathrm{FP}^n}$,
(c) $M(\pi_1, \pi_2) = \mathrm{FP}^n$.

Now, when we have spaces given as double mapping cylinders, we can use the following theorem to obtain embeddings.

THEOREM 4.2. *Retaining the previous notation let X be a compact, differentiable, n-dimensional manifold and assume we have maps h, T so that*

(i) $h: X \subsetneq R^l$ *with* $\eta_h = k\epsilon \oplus \bar{\eta}$ (where $\bar{\eta}$ is some subbundle of η_h and ϵ is the trivial line bundle),

(ii) $T: Z \subsetneq R^m$ *is a topological embedding,*

(iii) *there is a topological embedding $S: M(f) \to R^k \times R^m$ so that S restricted to Z is T, then there is a topological embedding of $M(f, g)$ in R^{l+m+1}.*

The proof is contained in [5]; it is similar to the proof of Theorem 1.2 of [9].

REMARK. When $M(f, g)$ is a manifold and we are in the metastable range then Haefliger's theorem [6] shows that we can assume the embedding is differentiable.

Now, using 4.1, 3.1 and 3.2 it is easy to complete the proofs of Theorems 2 and 3 exactly in the manner Theorem 1 is proved in [5].

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