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REPRESENTATIONS OF UNIFORMLY HYPERFINITE ALGEBRAS AND THEIR ASSOCIATED VON NEUMANNN RINGS

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Introduction. In this note we summarize the main results of a paper, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, which will be published elsewhere.

A uniformly hyperfinite (UHF) algebra of class $\{n_i\}$ is a C^*-algebra, $\mathfrak{A}$, which contains an increasing sequence of factors, $M_1 \subset M_2 \subset \cdots \subset \mathfrak{A}$, of types, $(\mathrm{I}_{n_1}), (\mathrm{I}_{n_2}), \cdots$, such that $\mathfrak{A}$ is the norm closure of $\bigcup_{i=1}^{\infty} M_i$. It is always assumed that the integers, $n_i \to \infty$ as $i \to \infty$. UHF algebras have been defined and studied by Glimm [2].

If Π is a *-representation of a UHF algebra, $\mathfrak{A}$, on a Hilbert space, then the von Neumann ring, $R = \{\Pi(\mathfrak{A})\}''$, generated by the representation algebra, $\Pi(\mathfrak{A})$, has the property that R is the strong closure of an increasing sequence of type (I_n) factors. Von Neumann rings with this property will be called hyperfinite rings. It is clear that every hyperfinite ring can be considered as the strong closure of a representation of a UHF algebra.

Murray and von Neumann showed that all hyperfinite factors of type (II_1) are isomorphic [**4**]. We consider the analogous question for type (III) hyperfinite factors and arrive at the conclusion that there are uncountably many nonisomorphic type (III) hyperfinite factors.

1. **Characterization of hyperfinite factors.** Hyperfinite factors may be characterized by states, positive linear functionals of norm one, of UHF algebras. If Π is a factor representation of a UHF algebra, $\mathfrak{A}$, on a Hilbert space, $\mathfrak{H}$, then any vector state, ω, of Π (i.e., any state of the form, $\omega(x)=(f, \Pi(x)f)$, for all $x\in\mathfrak{A}$, with $f\in\mathfrak{H}$, $\|f\|=1$) characterizes the representation, Π, up to quasi-equivalence. We recall that two representations, Π_1 and Π_2, of a C^*-algebra, $\mathfrak{A}$, are quasi-equivalent if and only if there is a *-isomorphism, ϕ, of the von Neumann ring, $\{\Pi_1(\mathfrak{A})\}''$, onto the von Neumann ring, $\{\Pi_2(\mathfrak{A})\}''$, such that $\phi(\Pi_1(x))=\Pi_2(x)$ for all $x\in\mathfrak{A}$ (see [**1**, §5.3]). It follows that if Π is a factor representation of a UHF algebra, $\mathfrak{A}$, and ω is a vector state of Π, then ω characterizes the hyperfinite factor, $\{\Pi(\mathfrak{A})\}''$, up to *-isomorphisms.

We consider the problem of determining the properties of those states of a UHF algebra, $\mathfrak{A}$, which induce a factor representations of $\mathfrak{A}$. It is convenient to introduce the following notation. If S is a subset of a C^*-algebra, $\mathfrak{A}$, we denote by S^c (the relative commutant of S in $\mathfrak{A}$) the subalgebra of $\mathfrak{A}$ consisting of all those $x\in\mathfrak{A}$ which commute with every element of S (i.e., $S^c=\{x\in\mathfrak{A};\, xy=yx \text{ for all } y\in S\}$). We find that factor states of a UHF algebra can be characterized by the following properties.

THEOREM 1.1. *Suppose* $\mathfrak{A}$ *is a* UHF *algebra and* $\{M_i;\, i=1, 2, \cdots\}$ *is an increasing sequence of type* (I_{n_i}) *factors which generate* $\mathfrak{A}$. *Suppose* ω *is a state of* $\mathfrak{A}$. *Then, the following conditions are equivalent.*

(i) *The state,* ω, *induces a factor representation of* $\mathfrak{A}$.

(ii) *For every* $x\in\mathfrak{A}$ *there is an integer,* $r>0$, *depending only on* x, *such that* $|\omega(xy)-\omega(x)\omega(y)|\leqq\|y\|$ *for all* $y\in M_r^c$.

(iii) *For every* $x\in\mathfrak{A}$ *there is a type* (I_n) *factor,* $M\subset\mathfrak{A}$, *such that* $|\omega(xy)-\omega(x)\omega(y)|\leqq\|y\|$ *for all* $y\in M^c$.

Using this result one can derive the following conditions that two factor representations of a UHF algebra be quasi-equivalent. If ω is a state of a C^*-algebra, $\mathfrak{A}$, and $M\subset\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$, we denote by $\omega|M$ the restriction of ω to M.

THEOREM 1.2. *Suppose* $\mathfrak{A}$ *is a* UHF *algebra and* $\{M_i;\, i=1, 2, \cdots\}$

is an increasing sequence of type (I_{n_i}) *factors which generate* $\mathfrak{A}$. *Suppose* Π_1 *and* Π_2 *are two factor representations of* $\mathfrak{A}$ *and* ω_1, ω_2 *are vector states of the representations,* Π_1 *and* Π_2, *respectively. Then, the following statements are equivalent.*

(i) Π_1 and Π_2 *are quasi-equivalent.*

(ii) *For every* $\epsilon > 0$ *there is an integer,* $i > 0$, *such that* $\|\omega_1 | M_i^c - \omega_2 | M_i^c\| < \epsilon$.

(iii) *For every* $\epsilon > 0$ *there is a type* (I_m) *factor,* $M \subset \mathfrak{A}$, *such that* $\|\omega_1 | M^c - \omega_2 | M^c\| < \epsilon$.

(iv) *There is a type* (I_n) *factor,* $N \subset \mathfrak{A}$, *such that* $\|\omega_1 | N^c - \omega_2 | N^c\| < 2$.

2. **Isomorphic hyperfinite rings.** We will say that two representations, Π_1 and Π_2, of a UHF algebra, $\mathfrak{A}$, are algebraically equivalent if the von Neumann rings, $\{\Pi_1(\mathfrak{A})\}''$ and $\{\Pi_2(\mathfrak{A})\}''$, are *-isomorphic. Clearly, any two irreducible representations of UHF algebra are algebraically equivalent. We obtain the following necessary and sufficient condition that two *-representations of a UHF algebra be algebraically equivalent.

THEOREM 2.1. *Suppose* Π_1 *and* Π_2 *are *-representations of a* UHF *algebra,* $\mathfrak{A}$, *on Hilbert spaces,* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *respectively. Suppose at least one of these Hilbert spaces is separable. Then, the representations,* Π_1 *and* Π_2, *are algebraically equivalent if and only if there is a *-automorphism,* ϕ, *of* $\mathfrak{A}$ *such that the representations,* $x \to \Pi_1(x)$ *and* $x \to \Pi(x) = \Pi_2(\phi(x))$, *are quasi-equivalent.*

This result can also be stated as follows. If Π_1 and Π_2 are algebraically equivalent representations of a UHF algebra, $\mathfrak{A}$, at least one of which is on a separable Hilbert space, then there is a *-isomorphism, ϕ, of $\{\Pi_1(\mathfrak{A})\}''$ onto $\{\Pi_2(\mathfrak{A})\}''$ such that ϕ maps $\Pi_1(\mathfrak{A})$ onto $\Pi_2(\mathfrak{A})$. Using this result one can show that the group of *-automorphisms of a UHF algebra, $\mathfrak{A}$, acts transitively on the pure state space of $\mathfrak{A}$. Combining Theorems 1.2 and 2.1 we obtain the following corollary.

COROLLARY 2.2. *Suppose* Π_1 *and* Π_2 *are factor representations of a* UHF *algebra,* $\mathfrak{A}$, *and* ω_1 *and* ω_2 *are vector states of* Π_1 *and* Π_2, *respectively. Then,* Π_1 *and* Π_2 *are algebraically equivalent if and only if there is a *-automorphism,* ϕ, *of* $\mathfrak{A}$ *such that the states,* $\omega_1(x)$ *and* $\omega(x) = \omega_2(\phi(x))$, *satisfy conditions* (ii), (iii) *and* (iv) *of Theorem* 1.2.

3. **Construction of nonisomorphic type** (III) **hyperfinite factors.** Suppose $\mathfrak{A}$ is a UHF algebra of class $\{2^i\}$. One can show that such an algebra contains and is generated by a sequence of mutually commutative factors, $\{N_r;\ r = 1, 2, \cdots\}$, of type (I_2). Each of these

factors, N_r, is spanned by a family of matrix units, $\{e_{ij}^{(r)}; i, j=1, 2; r=1, 2, \cdots\}$, characterized by the following properties, $e_{11}^{(r)}+e_{22}^{(r)}=I$, $e_{ij*}^{(r)}=e_{ji}^{(r)}$, $e_{ij}^{(r)}e_{mn}^{(r)}=\delta_{jm}e_{in}^{(r)}$, and $e_{ij}^{(r)}e_{nm}^{(s)}=e_{nm}^{(s)}e_{ij}^{(r)}$, for $r\neq s$, for $i, j, n, m=1, 2$ and $r, s=1, 2, \cdots$. We define a state, ω_λ, of $\mathfrak{A}$ for all $0\leqq\lambda\leqq\frac{1}{2}$ as the unique state of $\mathfrak{A}$, such that

$$\omega_\lambda(e_{i_1j_1}^{(r_1)}\cdots e_{i_nj_n}^{(r_n)}) = \lambda_{i_1}\delta_{i_1j_1}\cdots\lambda_{i_n}\delta_{i_nj_n},$$

for all $i_k, j_k=1, 2$; $r_k\neq r_n$ for $k\neq n$ and where $\lambda_1=\lambda$ and $\lambda_2=1-\lambda$. Since polynomials in the $\{e_{ij}^{(r)}\}$ are dense in $\mathfrak{A}$, it follows that the state, ω_λ, is uniquely determined. From Theorem 1.1 it follows that the states, ω_λ, induce factor representations, Π_λ, of $\mathfrak{A}$. The factors, $M_\lambda=\{\Pi_\lambda(\mathfrak{A})\}''$, have been studied by Glimm in [3]. In this paper Glimm shows that this family of factors, M_λ, is isomorphic to a family of factors constructed by Pukánszky [5] and, therefore, is able to show that for $0<\lambda<\frac{1}{2}$ the factors, M_λ, are of type (III). For $\lambda=0$, the state, ω_λ, is pure and, therefore, induces a type (I_∞) factor representation of $\mathfrak{A}$. For $\lambda=\frac{1}{2}$, the state, ω_λ, induces a type (II_1) factor representation of $\mathfrak{A}$. Using Corollary 2.2 we obtain the following result.

THEOREM 3.1. *The representations of $\mathfrak{A}$ induced by the states, ω_λ and $\omega_{\lambda'}$, are algebraically equivalent if and only if $\lambda=\lambda'$, i.e., the factors, M_λ, are mutually nonisomorphic for distinct λ, $0\leqq\lambda\leqq\frac{1}{2}$. Therefore, there are uncountably many nonisomorphic type* (III) *hyperfinite factors.*

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