ESSENTIAL SPECTRA OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS[1]

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Let A be a closed, densely defined operator in a Banach space X. There are several definitions of the "essential" spectrum of A (cf. [1], [2]). According to Wolf [3], [4] it is the complement in the complex plane of the Φ-set of A. The Φ-set Φ_A of A is the set of points λ for which

(a) $\alpha(A-\lambda)$, the dimension of the null space of $A-\lambda$, is finite

(b) $R(A-\lambda)$, the range of $A-\lambda$, is closed

(c) $\beta(A-\lambda)$, the codimension of $R(A-\lambda)$, is finite.

We denote the essential spectrum according to this definition by $\sigma_{ew}(A)$. The set $\sigma_{em}(A)$, as defined in [1], [2] is obtained by adding to $\sigma_{ew}(A)$ those points λ for which $\alpha(A-\lambda)\neq\beta(A-\lambda)$. It is the largest subset of $\sigma(A)$ which remains invariant under compact perturbations. Finally, to obtain the set $\sigma_{eb}(A)$, which is the essential spectrum according to Browder [5], we add to $\sigma_{em}(A)$ those points of $\sigma(A)$ which are not isolated.

Interest in the sets $\sigma_{ew}(A)$, $\sigma_{em}(A)$, $\sigma_{eb}(A)$ is centered about the fact that they remain invariant under certain perturbations of A. In particular one has

THEOREM 1. *Let A and B be closed densely defined operators in X. If $\lambda_0\in\rho(A)\cap\rho(B)$ and $(A-\lambda_0)^{-1}-(B-\lambda_0)^{-1}$ is a compact operator in X, then*

$$\sigma_{ew}(A) = \sigma_{ew}(B) \tag{1}$$

and

$$\sigma_{em}(A) = \sigma_{em}(B). \tag{2}$$

Moreover, if the complement $C\sigma_{em}(A)$ of $\sigma_{em}(A)$ is connected, then

$$\sigma_{eb}(A) = \sigma_{eb}(B). \tag{3}$$

This theorem was proved in [2] under the additional assumption that $D(B)\supseteq D(A)$. For selfadjoint operators the basic idea was employed by Birman [6], Wolf [4] and Rejto [7].

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We shall apply Theorem 1 to the situation of an elliptic operator perturbed by a potential. Let

$$A(x, D) = \sum_{|\mu|,|\nu|\leq r} D^\mu a_{\mu\nu}(x) D^\nu \tag{4}$$

be a uniformly strongly elliptic operator of order $2r$ defined in the whole of n-dimensional Euclidean space E^n. Here $\mu=(\mu_1, \cdots, \mu_n)$, $\nu=(\nu_1, \cdots, \nu_n)$ are multi-indices of nonnegative integers, $|\mu| =\mu_1+ \cdots +\mu_n$, $D^\mu=(-i)^{|\mu|}\partial^{|\mu|}/\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}$. The coefficients $a_{\mu\nu}(x)$ are to have bounded derivatives of all orders $\leq \max(|\mu|, |\nu|)$ in E^n and for $|\mu|=|\nu|=r$ the $a_{\mu\nu}(x)$ are to be uniformly continuous in E^n. By uniform strong ellipticity we mean that there is a constant $C_0>0$ such that

$$\operatorname{Re} \sum_{|\mu|=|\nu|=r} a_{\mu\nu}(x)\xi^\mu\xi^\nu \geqq C_0|\xi|^{2r}$$

for real vectors $\xi=(\xi_1, \cdots, \xi_n)$ and all $x\in E^n$, where $|\xi|^2=\xi_1^2+ \cdots +\xi_n^2$, $\xi^\mu=\xi_1^{\mu_1} \cdots \xi_n^{\mu_n}$.

Let A_0 be the operator $A(x, D)$ acting on the set C_0^∞ of infinitely differentiable functions with compact supports. We shall see that there is an extension of A_0 containing a half plane in its resolvent set.

Let A be a densely defined linear operator in a Hilbert space H. According to Kato [8] it is called *regularly accretive* if there is a bilinear form $a(u, v)$ such that

(1) $D(a)\supseteq D(A)$

(2) $\operatorname{Re} a$ is closed, and there is a constant $\gamma>0$ such that

$$\operatorname{Re} a(u, u) \geqq \gamma|\operatorname{Im} a(u, u)| \quad \text{for } u \in D(a) \tag{5}$$

(3) For $u\in D(a)$ and $f\in H$ one has

$$a(u, v) = (f, v) \quad \text{for all } v \in D(a) \tag{6}$$

if and only if $u\in D(A)$ and $Au=f$.

We call $a(u, v)$ a bilinear form if it is linear in u and conjugate linear in v. We write $a(u)$ in place of $a(u, u)$ and call $a(u, v)$ closed if $u_n\in D(a)$, $u_n\to u$ in H and $a(u_n-u_m)\to 0$ imply that $u\in D(a)$ and $a(u_n)\to a(u)$. It is called preclosed if $u_n\in D(a)$, $u_n\to 0$, $a(u_n-u_m)\to 0$ imply $a(u_n)\to 0$. It is easily seen that a preclosed nonnegative symmetric form has a closure (cf. [9]). The real and imaginary parts of a bilinear form are defined as

$$\operatorname{Re} a(u, v) = \tfrac{1}{2}[a(u, v) + \overline{a(v, u)}]; \quad \operatorname{Im} a(u,v) = \frac{1}{2i}[a(u, v) - \overline{a(v, u)}].$$

One can show easily that the bilinear form corresponding to a regularly accretive operator is unique. Let A be a regularly accretive extension of an operator A_0. We shall call A *minimal* if for any regularly accretive extension A' of A_0 we have

(i) $D(a) \subseteq D(a')$,

(ii) $a'(u, v) = a(u, v)$, $u, v \in D(a)$,

where a and a' are the bilinear forms corresponding to A and A', respectively.

The following theorem is basic in our study and answers a question raised by Kato [**8**].

THEOREM 2. *Let A_0 be a densely defined linear operator in H. Then a necessary and sufficient condition that A_0 have a regularly accretive extension is that*

$$\operatorname{Re}(A_0 u, u) \geqq \gamma \,|\operatorname{Im}(A_0 u, u)| \quad u \in D(A_0) \tag{7}$$

holds for some constant $\gamma > 0$. Moreover, there is a minimal extension which satisfies (7) *with the same constant γ.*

Returning to our operator $A(x, D)$ we wish to determine conditions on a function $q(x)$ so that $A_0 + q + \lambda$ has a regularly accretive extension. We formulate our conditions in terms of the following expressions (compare [**10**]). We define

$$\begin{aligned} M_{\alpha,p}(q) &= \sup_x \int_{|x-y|<1} |q(y)|^p |x-y|^\alpha dy \qquad -n < \alpha < 0 \\ &= \sup_x \int_{|x-y|<1} |q(y)|^p \left(1 + \log \frac{1}{|x-y|}\right) dy \qquad \alpha = 0 \\ &= \sup_x \int_{|x-y|<1} |q(y)|^p dy \qquad \alpha > 0 \end{aligned}$$

We let $M_{\alpha,p}$ be the set of functions q for which $M_{\alpha,p}(q) < \infty$.

For any function $h(x)$ we set $h^+(x) = \max[0, h(x)]$ and $h^-(x) = \min[0, h(x)]$. Employing Theorem 2, we have

THEOREM 3. *Assume that* $\operatorname{Im} q \in M_{2r-n,1}$ *and that* $(\operatorname{Re} q)^- \in M_{\alpha,1}$ *for some α satisfying $-n < \alpha < 2r - n$. Then there is a $\lambda_0 > 0$ such that $A_0 + q + \lambda$ has a regularly accretive extension for each $\lambda > \lambda_0$.*

We now apply Theorem 1 to obtain

THEOREM 4. *Let q satisfy the hypotheses of Theorem 3 and, in addition, assume that* $(\operatorname{Re} q)^+ \in M_{\beta,1}$, $\beta < 4r - n$, *and*

$$\int_{|x-y|<1} |q(y)|\,dy \to 0 \quad as\ |x| \to \infty. \tag{8}$$

Let A and B be the minimal regularly accretive extensions of $A_0+\lambda$ and $A_0+q+\lambda$, respectively, $\lambda>\lambda_0$. Then (1) *and* (2) *hold. If $C\sigma_{em}(A)$ is connected, then* (3) *holds.*

COROLLARY 5. *If* $\operatorname{Im} q \in M_{2r-n,1}$, $\operatorname{Re} q \in M_{\alpha,1}$ *and* (8) *holds, then the conclusions of Theorems* 3 *and* 4 *hold.*

Next we assume that there are constants $a_{\mu\nu}$ such that for each μ, ν, $|\mu|$, $|\nu| \leqq r$,

$$\int_{|x-y|<1} |a_{\mu\nu}(y) - a_{\mu\nu}|\,dy \to 0 \quad \text{as } |x| \to \infty. \tag{9}$$

We set

$$A(\infty, D) = \sum_{|\mu|,|\nu|\leq r} D^\mu a_{\mu\nu} D^\nu$$

and let A_∞ be the minimal operator of $A(\infty, D)$, i.e., the closure in L^2 of $A(\infty, D)$ defined on C_0^∞. The spectrum of $A_\infty+\lambda$ is easily computed via Fourier transforms and consists of the set R_λ of those complex η for which there is a real vector ξ satisfying $\sum_{|\mu|,|\nu|\leq r} a_{\mu\nu}\xi^\mu\xi^\nu+\lambda=\eta$. Moreover, if η is in $\sigma(A_\infty)$, then the range of $A_\infty-\eta$ is not closed in L^2. Hence we have

$$\sigma_{ew}(A_\infty) = \sigma_{em}(A_\infty) = \sigma_{eb}(A_\infty) = \sigma(A_\infty) = R_0. \tag{10}$$

We can now state

THEOREM 6. *If* (9) *holds as well as the assumptions of Theorems* 3 *and* 4, *we have*

$$\sigma_{ew}(B) = \sigma_{em}(B) = R_\lambda, \tag{11}$$

where B is the minimal regularly accretive extension of $A_0+q+\lambda$. If CR_λ is connected, we have

$$\sigma_{eb}(B) = R_\lambda \tag{12}$$

as well.

In the next two theorems we consider the perturbation of $A(x, D)$ by another operator of the same form

$$C(x, D) = \sum_{|\mu|,|\nu|\leq r} D^\mu C_{\mu\nu}(x) D^\nu.$$

We assume that $D^\mu(C_{\mu\nu}u)\in L^2$ for $u\in C_0^\infty$ and let C_0 denote the operator $C(x, D)$ with domain C_0^∞. We let $M_{-n,2}$ denote the set of essentially bounded functions and take α to satisfy $2r-n-2<\alpha<2r-n$. We make two sets of assumptions

H1. (a) $(\text{Re } C_{\mu\mu})^- \in M_{\alpha-2|\mu|,1}$ for $|\mu|\neq r$,
$(\text{Re } C_{\mu\mu})^- = 0$ for $|\mu| = r$

(b) For $\mu\neq\nu$

$$C_{\mu\nu}(x) + \bar{C}_{\nu\mu}(x) = g_{1\mu\nu}(x)h_{1\mu\nu}(x)$$

where

$$g_{1\mu\nu}(x) \in M_{2r-2|\mu|-n,2}, \quad h_{1\mu\nu}(x) \in M_{\alpha-2|\nu|,2} \quad |\nu| \neq r$$

$$g_{1\mu\nu}(x) \in M_{\alpha-2|\mu|,2}, \quad h_{1\mu\nu}(x) \in M_{-n,2}, \quad |\mu| \neq r, \quad |\nu| = r$$

$$g_{1\mu\nu}(x) \equiv 0 \quad |\mu| = |\nu| = r$$

(c) $C_{\mu\nu}(x) - \bar{C}_{\nu\mu}(x) = g_{2\mu\nu}(x)h_{2\mu\nu}(x)$

where

$$g_{2\mu\nu}(x) \in M_{2r-2|\mu|-n,2}, \quad h_{2\mu\nu}(x) \in M_{2r-2|\nu|-n,2}$$

H2. (a) $(\text{Re } C_{\mu\mu})^+ \in M_{\beta-2|\mu|,1}$, $\beta<4r-n$

(b) $\int_{|x-y|<1} |C_{\mu\nu}(y)|\,dy \to 0$ as $|x|\to\infty$.

THEOREM 7. *Under assumptions* H1 *there is a* $\lambda_0>0$ *such that* $A_0+C_0+\lambda$ *has a regularly accretive extension for each* $\lambda>\lambda_0$.

THEOREM 8. *Under assumptions* H1 *and* H2, *we have*

$$\sigma_{ew}(E) = \sigma_{ew}(A), \qquad \sigma_{em}(E) = \sigma_{em}(A), \tag{13}$$

where E is the minimal regularly accretive extension of $A_0+C_0+\lambda$. *If* $C\sigma_{em}(A)$ *is connected, then* $\sigma_{eb}(E)=\sigma_{eb}(A)$. *If* (9) *holds, then*

$$\sigma_{ew}(E) = \sigma_{em}(E) = R_\lambda. \tag{14}$$

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REPRESENTATIONS OF UNIFORMLY HYPERFINITE ALGEBRAS AND THEIR ASSOCIATED VON NEUMANNN RINGS

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Introduction. In this note we summarize the main results of a paper, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, which will be published elsewhere.

A uniformly hyperfinite (UHF) algebra of class $\{n_i\}$ is a C^*-algebra, $\mathfrak{A}$, which contains an increasing sequence of factors, $M_1 \subset M_2 \subset \cdots \subset \mathfrak{A}$, of types, (I_{n_1}), (I_{n_2}), $\cdots$, such that $\mathfrak{A}$ is the norm closure of $\bigcup_{i=1}^{\infty} M_i$. It is always assumed that the integers, $n_i \rightarrow \infty$ as $i \rightarrow \infty$. UHF algebras have been defined and studied by Glimm [2].

If Π is a *-representation of a UHF algebra, $\mathfrak{A}$, on a Hilbert space, then the von Neumann ring, $R = \{\Pi(\mathfrak{A})\}''$, generated by the representation algebra, $\Pi(\mathfrak{A})$, has the property that R is the strong closure of an increasing sequence of type (I_n) factors. Von Neumann rings with this property will be called hyperfinite rings. It is clear that every