CONSTRUCTIVE PROOF OF THE EXISTENCE OF MULTIPLICATIVE FUNCTIONALS IN COMMUTATIVE SEPARABLE BANACH ALGEBRAS

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Gelfand's 1941 proof of the existence of multiplicative functionals in commutative Banach algebras is essentially based on Zorn's axiom.

In 1961, P. J. Cohen [3] gave a constructive (i.e. free from Zorn's axiom) way to get rid of Banach algebras in some of their applications.

This year, E. Bishop [1], [2] has presented a theory of Banach algebras in the frame of L. E. J. Brouwer's constructivist ideas. Therefrom it is easy to deduce a constructive proof of the existence of multiplicative functionals. However this proof would be needlessly intricate when just interested in constructive methods.

Here is a simple constructive proof of Gelfand's theorem.

1. Let A be a commutative separable Banach algebra with unit 1 throughout the paper.

Let us recall some properties of ideals of A.

(a) *0, A, $\sum_{i=1}^{m} x_i A$ and $\mathfrak{J}+\sum_{i=1}^{m} x_i A$ are ideals of A whenever $x_1, \cdots, x_m \in A$ and $\mathfrak{J}$ is an ideal of A.*

(b) *If an ideal $\mathfrak{J}$ contains an invertible element, then $\mathfrak{J}=A$.*

(c) *Let $\mathfrak{J}\neq A$ be an ideal, then $d[1, \mathfrak{J}]=1$.*

Since $0\in\mathfrak{J}$, $d[1, \mathfrak{J}]\leqq 1$. Moreover if $d[1, \mathfrak{J}]<1$, there exists $x_0\in\mathfrak{J}$ such that $d[1, x_0]<1$. Then x_0^{-1} exists and consequently $1=x_0x_0^{-1}$ belongs to $\mathfrak{J}$.

(d) *Let $\mathfrak{J}\neq A$ be an ideal. If $1-xy\in\mathfrak{J}$, then $d[x, \mathfrak{J}]\geqq 1/\|y\|$.*

In fact, $\mathfrak{J}\neq A$ implies $d[1, \mathfrak{J}]=1$ and since $1-xy\in\mathfrak{J}$, we have $d[xy, \mathfrak{J}]=1$ and $d[xy, \mathfrak{J}]\leqq d[xy, y\mathfrak{J}]\leqq\|y\| d[x, \mathfrak{J}]$.

2. We need a lemma, which is a direct version of the classical fact that the spectrum of the Banach algebra E/A is not void.

Let $\mathfrak{J}\neq A$ be an ideal. Then for all $x\in A$, there exists $z\in \mathbf{C}$ such that $\mathfrak{J}+(x-z)A\neq A$.

Suppose there exists an ideal $\mathfrak{J}\neq A$ and $x\in A$ such that $\mathfrak{J}+(x-z)A=A$ for all $z\in\mathbf{C}$.

Then for all $z\in\mathbf{C}$, there is at least one element $a(z)\in A$ with $1-(x-z)a(z)\in\mathfrak{J}$.

Let $\mathfrak{x}$ be any continuous linear functional in A vanishing on $\mathfrak{J}$.

(a) *$\mathfrak{x}[a(z)]$ depends only on $z\in\mathbf{C}$ and not on the choice of $a(z)$.*

In fact if $1-(x-z)a_i \in \mathfrak{J}$, $(i=1, 2)$, then

$$\mathfrak{x}[a_1 - a_2] = \mathfrak{x}[(1 - (x - z)a_2)a_1 - (1 - (x - z)a_1)a_2] = 0.$$

So $\mathfrak{x}[a(z)]$ is defined without using the axiom of choice.

(b) *$\mathfrak{x}[a(z)]$ is holomorphic on $\mathbf{C}$.*

In the neighborhood $V(z_0) = \{z: |z-z_0| < 1/\|a(z_0)\|\}$ of $z_0 \in \mathbf{C}$, $1-(z-z_0)a(z_0)$ is invertible and $\mathfrak{x}[a(z)] = \mathfrak{x}[a(z_0)/(1-(z-z_0)a(z_0))]$ since

$$a(z) - \frac{a(z_0)}{1 - (z - z_0)a(z_0)} = \frac{[1 - (x - z_0)a(z_0)]a(z) - [1 - (x - z)a(z)]a(z_0)}{1 - (z - z_0)a(z_0)} \in \mathfrak{J}.$$

(c) *In fact,* $\mathfrak{x}[a(z)] \equiv 0$.

For $|z| > \|x\|$, $x-z$ is invertible and $\mathfrak{x}[a(z)] = \mathfrak{x}[(x-z)^{-1}]$ because

$$(x - z)^{-1} - a(z) = (x - z)^{-1}[1 - (x - z)a(z)] \in \mathfrak{J}.$$

Hence the conclusion by Liouville's theorem, since $\mathfrak{x}[(x-z)^{-1}] \to 0$ when $z \to \infty$, from the inequalities

$$|\mathfrak{x}[(x - z)^{-1}]| \leqq C\|(x - z)^{-1}\| \leqq C(|z| - \|x\|)^{-1}.$$

(d) *There exists $\mathfrak{x}$ vanishing on $\mathfrak{J}$ and such that $\mathfrak{x}[a(z)] \neq 0$.*

Let us fix $z_0 \in \mathbf{C}$. Since $d[1, \mathfrak{J}] = 1$ and $1-(x-z_0)a(z_0) \in \mathfrak{J}$, we have $d[a(z_0), \mathfrak{J}] \geqq 1/(\|x\| + |z_0|)$. As A is separable, by Hahn-Banach's theorem (see [**4**], for instance), there is a continuous linear functional $\mathfrak{x}$ such that

$$\mathfrak{x}[a(z_0)] = 1/(\|x\| + |z_0|), \quad \mathfrak{x}[\mathfrak{J}] = 0.$$

So (c) and (d) are contradictory, hence the lemma.

3. Let us prove Gelfand's theorem.

Let A be a commutative separable Banach algebra with unit 1.

If $\mathfrak{J} \neq A$ is an ideal, then there exists a continuous nonzero multiplicative functional vanishing on $\mathfrak{J}$.

Let x_m be a dense sequence in A.

Since $\mathfrak{J} \neq A$, by successive applications of the lemma we get a sequence $z_m \in \mathbf{C}$ such that $\mathfrak{J} + \sum_{i=1}^{m} (x_i - z_i)A \neq A$, hence

$$d\left[1, \mathfrak{J} + \sum_{i=1}^{m} (x_i - z_i)A\right] = 1, \quad \forall m.$$

Therefore there exists a sequence $\mathfrak{x}_m$ of continuous linear functionals such that

$$\|\mathfrak{x}_m\| = 1, \quad \mathfrak{x}_m(1) = 1, \quad \mathfrak{x}_m\left[\mathfrak{J} + \sum_{i=1}^{m} (x_i - z_i)A\right] = 0.$$

Hence we get

$$\mathfrak{x}_m(\mathfrak{J}) = 0, \ \mathfrak{x}_m(x_i) = z_i,$$

$$\mathfrak{x}_m(x_i x_j) = z_i \mathfrak{x}_m(x_j) = \mathfrak{x}_m(x_i)\mathfrak{x}_m(x_j), \quad \forall i, j \leqq m$$

for

$$\mathfrak{J}, \ x_i - z_i, \ x_i x_j - z_i x_j \in \mathfrak{J} + \sum_{i=1}^{m} (x_i - z_i)A.$$

As $\|\mathfrak{x}_m\| = 1$ for all m, there is a weak convergent subsequence of $\mathfrak{x}_m$ for A is separable (see [4], for instance). Let $\mathfrak{x}$ be its limit.

Of course $\mathfrak{x}$ is a continuous linear functional and

$$\|\mathfrak{x}\| = 1, \qquad \mathfrak{x}(1) = 1, \qquad \mathfrak{x}(\mathfrak{J}) = 0.$$

Moreover $\mathfrak{x}$ is a multiplicative functional. In fact, we have

$$\mathfrak{x}(x_i x_j) = \mathfrak{x}(x_i)\mathfrak{x}(x_j), \quad \forall i, j$$

and the sequence x_m is dense in A.

References

1. E. Bishop, Seminar in La Jolla, 1965–1966.

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