THE DIFFEOMORPHISM GROUP OF A COMPACT RIEMANN SURFACE

BY C. J. EARLE[1] AND J. EELLS

Communicated by S. Smale, February 6, 1967

1. **Introduction.** In this note we announce two theorems. The first describes the homotopy type of the topological group $\mathfrak{D}(X)$ of diffeomorphisms ($=C^\infty$-diffeomorphisms) of a compact oriented surface X without boundary. The second, of which the first is a corollary, gives a fundamental relation among $\mathfrak{D}(X)$, the space of complex structures on X, and the Teichmüller space $T(X)$ of X. We make essential use of the theory of quasiconformal mappings and Teichmüller spaces developed by Ahlfors and Bers [3], [6], and the theory of fibrations of function spaces. Our results confirm a conjecture of Grothendieck [7, p. 7-09], relating the homotopy of $\mathfrak{D}(X)$ and $T(X)$.

2. **The theorems.** The surface X has a unique (up to equivalence) C^∞-differential structure. Let $\mathfrak{D}(X)$ denote the group of orientation preserving diffeomorphisms. With the C^∞-topology (uniform convergence of all differentials) $\mathfrak{D}(X)$ is a metrizable topological group [8]. We let $\mathfrak{D}_0(X; x_1, \cdots, x_n)$ denote the subgroup of $\mathfrak{D}(X)$ consisting of those diffeomorphisms f which are homotopic to the identity and satisfy $f(x_i)=x_i$ $(1 \leqq i \leqq n)$, where $x_1, \cdots, x_n$ are distinct points of X. This second condition is fulfilled vacuously if $n=0$.

THEOREM 1. *Let g denote the genus of X.*

(a) *If $g=0$, then $\mathfrak{D}_0(X; x_1, x_2, x_3)$ is contractible. Furthermore, $\mathfrak{D}(X)$ is homeomorphic to $G \times \mathfrak{D}_0(X; x_1, x_2, x_3)$, where G is the group of conformal automorphisms of the Riemann sphere.*

(b) *If $g=1$, then $\mathfrak{D}_0(X; x_1)$ is contractible. Furthermore, $\mathfrak{D}_0(X)$ is homeomorphic to $G \times \mathfrak{D}_0(X; x_1)$, where now G is the identity component of the group of conformal automorphisms of the torus.*

(c) *If $g \geqq 2$, then $\mathfrak{D}_0(X)$ is contractible.*

COROLLARY. *In all cases $\mathfrak{D}_0(X)$ is the identity component of $\mathfrak{D}(X)$.*

REMARK 1. Part (a) is equivalent to the theorem of Smale [9] asserting that the rotation group SO(3) is a strong deformation retract of $\mathfrak{D}(S^2)$. Our proof is entirely different from Smale's.

REMARK 2. A concept of differentiability has recently been de-

[1] Research partially supported by NSF Grant GP6145.

veloped by J. A. Leslie, relative to which he has shown that the group $\mathfrak{D}(X)$—for any compact differential manifold—is a Lie group [8].

Next, let $M(X)$ denote the C^∞-complex structures on X; with the C^∞-topology this is a convex open subset of a Fréchet space. Any subgroup of $\mathfrak{D}(X)$ operates (on the right) on $M(X)$ in the obvious way. If $g \geqq 1$ let $T(X)$ denote the Teichmüller space of X. For the definition of that space we refer to [2], [6, Lecture 1]; however, Theorem 2 below provides a new characterization.

THEOREM 2. *Let g denote the genus of X.*

(a) *If $g=0$, then $\mathfrak{D}_0(X; x_1, x_2, x_3)$ is homeomorphic to $M(X)$.*

(b) *If $g=1$, then $\mathfrak{D}_0(X; x_1)$ operates principally (i.e., continuously, freely, properly, and with local sections) on $M(X)$. The quotient space is (homeomorphic to) $T(X)$.*

(c) *If $g \geqq 2$, then $\mathfrak{D}_0(X)$ operates principally on $M(X)$. The quotient space is (homeomorphic to) $T(X)$.*

3. **On the proof of Theorem 1.** We proceed to indicate how Theorem 1 is derived from Theorem 2.

First of all, $M(X)$ is always contractible. Therefore Theorem 2a easily implies Theorem 1a. Secondly, the quotient map $M(X) \rightarrow T(X)$ defines a principal fibre bundle. A fundamental theorem of Teichmüller (see [1], [5] for efficient proofs) asserts that $T(X)$ is a finite dimensional cell. Thus from the homotopy sequence of our fibration we conclude that all the homotopy groups of the structural groups $\mathfrak{D}_0(X; x_1)$ and $\mathfrak{D}_0(X)$ vanish. Finally, these structural groups are metrizable manifolds modeled on Fréchet spaces; therefore, they are absolute neighborhood retracts. By a theorem of J. H. C. Whitehead the vanishing of their homotopy groups implies their contractibility. Theorem 1 follows.

4. **On the proof of Theorem 2.** For simplicity of exposition we consider only the cases $g \geqq 2$. We represent X as the quotient of the upper half plane U by a Fuchsian group Γ operating freely on U. The C^∞ complex structures ($=C^\infty$ Beltrami differentials [2]) on X are represented by the C^∞ complex valued functions μ on U satisfying

$$\mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$$

and $\max\{|\mu(z)| : z \in U\} < 1$. Each such function μ determines uniquely a diffeomorphism f: $U \rightarrow U$ such that

$$\mu(z) = \mu_f(z) = f_{\bar{z}}(z)/f_z(z),$$

and f leaves the points 0, 1, ∞ fixed.

The group $\mathfrak{D}_0(X)$ is identified with the group of diffeomorphisms $h: U \to U$ which commute with all elements of Γ. The action of $\mathfrak{D}_0(X)$ on $M(X)$ is given by

$$\mu_f \cdot h = \mu_{f \circ h}.$$

This action is principal, and the quotient $M(X)/\mathfrak{D}_0(X)$ is homeomorphic to $T(X)$. For the verification we first study the dependence on μ of the solutions of Beltrami's equation $f_{\bar{z}} = \mu f_z$. Next we verify that $M(X)/\mathfrak{D}_0(X)$ maps bijectively and continuously onto $T(X)$. Finally, a theorem of Ahlfors and Weill [4] provides us with local sections from $T(X)$ into $M(X)$.

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Cornell University and
Churchill College, Cambridge