DUALITY METHODS AND PERTURBATION OF SEMIGROUPS

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1. **Introduction.** In [5], the author announced several theorems applying the semi-inner product methods of Lumer and Phillips [4] to the perturbation theory of one-parameter holomorphic contraction semigroups on Banach spaces. This note extends the methods to a perturbation theorem of Trotter [9], with proofs, and announces generalizations to locally convex spaces. (See also Kato [3].)

2. **Generation theorem, ϕ-sectorial operators.** Let $\mathfrak{X}$ be a complex Banach space, and let $[\ ,\ ]: \mathfrak{X}\times\mathfrak{X}\to\mathbf{C}$ be a semi-inner product for $\mathfrak{X}$, in the sense of [4]: (i) for all $v\in\mathfrak{X}$, $u\to[u, v]$ is a linear functional on $\mathfrak{X}$, (ii) $[u, u]\geqq 0$ for all $u\in\mathfrak{X}$, with $\|u\|=[u, u]^{1/2}$, and (iii) $|[u, v]| <\|u\|\,\|v\|$.

DEFINITION 1. A linear operator A with domain $\mathfrak{D}(A)\subset\mathfrak{X}$ is *ϕ-sectorial* for $0\leqq\phi\leqq\pi/2$ iff for every $u\in\mathfrak{D}(A)$,

$$\tan\phi\,|\operatorname{Im}[Au, u]| \leqq -\operatorname{Re}[Au, u] \geqq 0. \tag{1}$$

Every ϕ_1-sectorial operator is ϕ_2-sectorial for all $\phi_2\leqq\phi_1$, and every 0-sectorial operator is dissipative (Re $[Au, u]\leqq 0$ as in [4]). If $\phi=\pi/2$, replace the first inequality by Im $[Au, u]=0$. If $\Delta_\phi=\{z\,|\,\pi\geqq|\arg z|\geqq\pi/2+\phi\}$, and $W(A)=\{[Au, u]\,|\,u\in\mathfrak{D}(A), \|u\|=1\}$ is the numerical range of A then A is ϕ-sectorial iff $\Delta_\phi\supset\{W(A)\}^-$ (obvious when sketched).

DEFINITION 2. A one-parameter semigroup T is in the family $CH(\phi)$ of *holomorphic contraction semigroups* on the sector $S_\phi=\{z\,|\,|\arg z|\leqq\phi\}$ iff

(a) T is a homomorphism of the additive semigroup of S_ϕ into the multiplicative semigroup $\mathfrak{C}(\mathfrak{X})$ of all contraction operators on $\mathfrak{X}$ ($\|T(z)\|\leqq 1$),

(b) $z\to T(z)$ is a holomorphic function from $\operatorname{int}(S_\phi)$ to $\mathfrak{C}(\mathfrak{X})\subset\mathfrak{L}(\mathfrak{X})$, the Banach algebra of bounded operators on $\mathfrak{X}$ (see [2, Chapter 5]), and

(c) (slightly redundant) for all $u\in\mathfrak{X}$, the map $z\to T(z)u$ is continuous from S_ϕ into $\mathfrak{X}$.

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(d) Then the (infinitesimal) generator A of T is defined, for all $u\in \mathfrak{X}$ where the limit through real h exists, by

$$Au = \lim_{h\searrow 0} h^{-1}\{T(h)u - u\}. \tag{2}$$

THEOREM 1. *An operator A is the infinitesimal generator of a $CH(\phi)$ semigroup T iff*

(a) *A is closed, densely defined, and ϕ-sectorial, and*

(b) *$\rho(A)\cap(\mathbf{C}\sim\Delta_\phi)\neq\varnothing$, where $\rho(A)$ is the resolvent set.*

LEMMA A. *A is ϕ-sectorial iff $e^{i\theta}A$ is dissipative for all $0\leqq|\theta|\leqq\phi\leqq\pi/2$.*

PROOF.

$$\begin{aligned}\operatorname{Re}[e^{i\theta}Au, u] &= \operatorname{Re}(e^{i\theta}[Au, u]) \\ &= \cos\theta\operatorname{Re}[Au, u] - \sin\theta\operatorname{Im}[Au, u].\end{aligned}\tag{3}$$

Since all such $\cos\theta$ are positive, $e^{i\theta}A$ is dissipative and (3) is negative for all $u\in\mathfrak{D}(A)$ iff, dividing by $-\cos\theta$,

$$0\leqq -\operatorname{Re}[Au, u] \geqq \operatorname{Im}[Au, u]\tan\theta. \tag{4}$$

Since $\tan\theta$ is monotone increasing, this holds for all θ in the specified range iff it holds for $\theta=\pm\phi$, depending upon the sign of $\operatorname{Im}[Au, u]$. This last is equivalent to (1).

LEMMA B. *Suppose A is ϕ-sectorial, $u\in\mathfrak{D}(A)$, and $z\notin\Delta_\phi$. Then*

$$\|(z - A)u\| \geqq d(z, \Delta_\phi)u, \tag{5}$$

where $d(z, \Delta_\phi)$ is the distance from z to Δ_ϕ.

PROOF. Clearly if $|\arg z|\leqq\phi$, $d(z, \Delta_\phi)=|z|$. Then

$$\begin{aligned}\|(z - A)u\| &= \|\exp(i\arg z)(|z| - \exp(-i\arg z)A)u\| \\ &= \|(|z| - \exp(-i\arg z)A)u\|.\end{aligned}\tag{6}$$

But here $\exp(-i\arg z)A$ is dissipative by Lemma 2, and a calculation from [4] yields for any dissipative B and $z_0\in\mathbf{C}$:

$$\begin{aligned}\operatorname{Re}(z_0)\|u\|^2 = \operatorname{Re}[z_0u, u] &\leqq \operatorname{Re}([z_0u, u] - [Bu, u]) \\ &\leqq |[(z_0 - B)u, u]| \leqq \|(z_0 - B)u\|\,\|u\|.\end{aligned}\tag{7}$$

Cancelling $\|u\|$ and applying this with $z_0=|z|$, $B=\exp(-i\arg z)A$,

$$\|(z - A)u\| \geqq \operatorname{Re}(z_0)\|u\| = |z|\,\|u\| = d(z, \Delta_\phi)\|u\| \tag{8}$$

by (6).

But if $\phi \leqq |\arg z| \leqq \pi - \phi$, trigonometry shows that

$$d(z, \Delta_\phi) = \cos(\arg z - \phi)\,|z|. \tag{9}$$

Then $e^{-i\phi}A$ is dissipative by Lemma 2 and by the same procedure with $z_0 = \exp(i(\arg z - \phi))|z|$, $B = e^{-i\phi}A$,

$$\begin{aligned} \|(z - A)u\| &= \|e^{i\phi}\{\exp(i(\arg z - \phi))\,|z| - e^{-i\phi}A\}u\| \\ &= \|(\exp(i(\arg z - \phi))\,|z| - e^{-i\phi}A)u\| \\ &\geqq \operatorname{Re}(\exp(i(\arg z - \phi))\,|z|)\|u\| \\ &= \cos(\arg z - \phi)\,|z|\,\|u\| = d(z, \Delta_\phi)\|u\|. \end{aligned} \tag{10}$$

LEMMA C. *If (5) holds and A is closed $\rho(A)$ either is disjoint from $\mathbf{C}\sim\Delta_\phi$ or contains $\mathbf{C}\sim\Delta_\phi$ (the complement of Δ_ϕ in $\mathbf{C}$).*

PROOF. If $z_0 \in \rho(A) \cap (\mathbf{C}\sim\Delta_\phi)$ then (5) yields

$$\|(z_0 - A)^{-1}\| \leqq d(z_0, \Delta_\phi)^{-1}. \tag{11}$$

Then the argument of [**10**, Theorem VIII.2.1], with this estimate, shows that for z in the open disc about z_0 tangent to Δ_ϕ ($|z_0 - z| < d(z_0, \Delta_\phi)$), $z \in \rho(A)$, with the Neumann expansion

$$(z - A)^{-1} = \sum_{k=0}^{\infty} (z_0 - z)^k (z_0 - A)^{-(k+1)}. \tag{12}$$

Any nonempty subset of $\mathbf{C}\sim\Delta_\phi$ containing a Δ_ϕ-tangent disc about each of its members exhausts $\mathbf{C}\sim\Delta_\phi$ (induction).

LEMMA D (HILLE). *Let T be any strongly continuous semigroup on S_ϕ whose restriction to* $\operatorname{int}(S_\phi)$ *is in $H(\phi, \phi)$* ([**2**, *Definition* 10.6.1, p. 325]), *and whose generator is A. Then for $|\theta| \leqq \phi$, $t \to T_\theta(t) = T(e^{i\theta}t)$ is a semigroup of class C_0 with generator $A_\theta = e^{i\theta}A$. If $T \in CH(\phi)$, then $T_\theta \in CH(0)$ and $e^{i\theta}A$ is dissipative for $|\theta| \leqq \phi$.*

PROOF. The argument of Lemma 10.6.2 and the first part of the proof of Theorem 12.8.1 in [**2**] yields finite constants M_ϕ and ω_ϕ with

$$\|T(e^{i\theta}t)\| \leqq M_\phi e^{\omega t}. \tag{13}$$

Then the deformation-of-contours argument on page 384 of [**2**] leads to the following rewording of 12.8.4:

$$(\lambda - A)^{-1}u = e^{i\theta}\int_0^\infty e^{-\lambda e^{i\theta}t}T(e^{i\theta}t)u\,dt = e^{i\theta}(e^{i\theta}\lambda - A_\theta)^{-1}u \equiv v. \tag{14}$$

For such a v (these exhaust $\mathfrak{D}(A) = \mathfrak{D}(A_\theta)$)

$$
\begin{aligned}
e^{i\theta}Av = -e^{i\theta}(\lambda - A)v + e^{i\theta}\lambda v &= -e^{i\theta}u + e^{i\theta}\lambda v \\
&= -e^{i\theta}u + (e^{i\theta}\lambda - A_\theta)v + A_\theta v \\
&= -e^{i\theta}u + e^{i\theta}u + A_\theta v = A_\theta v,
\end{aligned} \tag{15}
$$

substituting twice and cancelling.

If T consists entirely of contractions, so does T_θ, hence Theorem 3.2 of [4] applies to prove that $e^{i\theta}A = A_\theta$ is dissipative.

PROOF OF THEOREM 1. If $T \in CH(\phi)$, every $e^{i\theta}A$ is dissipative by Lemma D, for $|\theta| \leqq \phi$, so A is ϕ-sectorial by Lemma A. By Theorem 3.2 of [4] again, since T_0 is a contraction semigroup, $A_0 = A$ is closed, densely defined, and has $1 \in \rho(A) \cap \mathbf{C} \sim \Delta_\phi$. Hence (a) and (b) hold.

Suppose (a) (especially A ϕ-sectorial) and (b), so that by Lemmas B and C and equation (5),

$$
\|(z - A)^{-1}\| \leqq d(z, \Delta_\phi)^{-1}. \tag{16}
$$

Then Hille's Theorem 12.8.1 of [2] shows that A generates a $H(-\phi, \phi)$ semigroup T^h on $\operatorname{int}(S_\phi)$. Applying Lemma D to closed subsectors, and Lemma 1 to see that $e^{i\theta}A$ is dissipative, it follows that all $T^h(e^{i\theta}t)$ are contractions by Theorem 3.2 of [4]. It remains to show that the contraction semigroups generated by $e^{\pm i\phi}A$ extend T^h to all of S_ϕ, forming a $CH(\phi)$ semigroup T. All T_θ generated by $e^{i\theta}A$ for $|\theta| \leqq \phi$ leave the common $\mathfrak{D}(A) = \mathfrak{D}(e^{i\theta}A)$ invariant, are differentiable on it and commute with A (Theorem 10.3.3 of [2]). If $u \in \mathfrak{D}(A)$,

$$
\begin{aligned}
T_{\pm\phi}(t)u - T_\theta(t)u &= \int_0^t \frac{d}{ds}(T_{\pm\phi}(s)T_\theta(t-s)u)ds \\
&= \int_0^t T_{\pm\phi}(s)(e^{\pm i\phi}A - e^{i\theta}A)T_\theta(t-s)uds \\
&= \{e^{\pm i\phi} - e^{i\theta}\}\int_0^t T_{\pm\phi}(s)T_\theta(t-s)Auds.
\end{aligned} \tag{17}
$$

Since the T's are contractions, the last integral is smaller than $t\|Au\|$ and, as $\theta \to \pm\phi$, $T_\theta(t)u \to T_{\pm\phi}(t)u$ uniformly on t-compacta, allowing a continuous extension of $z \to T^h(z)u$ to S_ϕ. By $3-\epsilon$, this extends to all $u \in X$, and the semigroup property extends by limits as well, to create a $T \in CH(\phi)$.

3. The perturbation theorems.

THEOREM 2. (a) *If A and B are ϕ-sectorial, and α and β nonnegative, then $D = \alpha A + \beta B$ is ϕ-sectorial* (see [8]).

(b) *If* $\{A_\alpha | \alpha \in I\}$ *is a net of* ϕ*-sectorial operators, and D is defined, for all u where the limit exists in* $\bigcap \{\mathfrak{D}(A_\alpha) | \alpha \in I\}$, *by* $Du = \lim A_\alpha u$, *then D is* ϕ*-sectorial.*

PROOF. (a) If $u \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$, $[(\alpha A + \beta B)u, u] = \alpha[Au, u] + \beta[Bu, u]$. Then $W(\alpha A + \beta B) \subset \alpha W(A) + \beta W(B) \subset \Delta_\phi$ since Δ_ϕ is a cone; the same applies for closures since Δ_ϕ is closed (Def. 1 *et seq.*).

(b) $[Du, u] = [(D - A_\alpha)u, u] + [A_\alpha u, u]$ and $|[(D - A_\alpha)u, u]| \leq \|(D - A_\alpha)u\| \|u\| \to 0$; so $[Du, u] = \lim [A_\alpha u, u]$, and the same applies to real and imaginary parts, so (1) for D follows from (1) for the A_α.

THEOREM 3. *Suppose D in Theorem* 2 (a) *or* (b) *is densely defined, and for some* $z_0 \notin \Delta_\phi$, *range* $(z - D)$ *is dense. Then* $\overline{D}$ *exists and generates a* $CH(\phi)$ *semigroup.*

PROOF. All $e^{i\theta}D$ for $|\theta| \leq \phi$ are dissipative by Lemma A. Theorem 3.3 of [4] insures that $\overline{D}$ exists, and an easy modification of the proof of their Lemma 3.4 shows that a new semi-inner product can be chosen making all $e^{i\theta}\overline{D}$ dissipative at once, so $\overline{D}$ becomes ϕ-sectorial. Then as in Theorem VIII.1.1 of [10, p. 209], $z_0 \in \rho(\overline{D})$ and Theorem 1 applies.

The following can supply the range condition:

(DA) D has a densely defined dissipative adjoint D^*; e.g. in (b) the net $\{A_\alpha^* | \alpha \in I\}$ consists of dissipative operators converging on a dense subset of $\mathfrak{X}^*$ (see Corollary 3.2 in [4]).

(G) In (a), $\mathfrak{D}(A) \subset \mathfrak{D}(B)$ and for some $a < 1$ and $b \geq 0$, $\|Bu\| \leq a\|Au\| + b\|u\|$ for all $u \in \mathfrak{D}(A)$ (see [1]).

THEOREM 4. *If* $\{A_\alpha | \alpha \in I\}$ *is a net of generators of* $CH(\phi)$ *semigroups* T_α *with a limit D satisfying Theorem* 3 (*or* (DA)) *then* $\overline{D}$ *generates a* $CH(\phi)$ *semigroup T which is the uniform strong limit on compacta in* S_ϕ *of the* T_α.

PROOF. We already know that T exists and that $C \sim \Delta_\phi \subset \rho(\overline{D})$. The usual argument for the Trotter-Kato theorem (see [10, p. 270–271]) can then be shortened considerably because the limit semigroup T and limit resolvents $(z - \overline{D})^{-1}$ are already known to exist, but essentially the same reasoning is used to prove uniform convergence on compacta. (The novelty lies in the treatment of the cases $\phi \neq 0$ and the avoidance of ergodic theorems for pseudoresolvents. For another proof, see [7].)

4. Generalizations. If a family Γ of seminorms p *calibrates* (gives a locally convex topology to) a complex vector space $\mathfrak{X}$, there is a

Lumer structure $\Lambda = \{[\ ,\ ]_p \mid p \in \Gamma\}$ for $\mathfrak{X}$ consisting of indefinite semi-inner products with $[u, u]_p^{1/2} = p(u)$. A is ϕ-sectorial for Λ if it is ϕ-sectorial for every $[\ ,\ ]_p$, and $T(z)$ is a *contraction* iff for all $u \in \mathfrak{X}$ and $p \in \Gamma$, $p(T(z)u) \leqq p(u)$. If "holomorphic" is taken to mean "$z \to \langle u^*, T(z)u\rangle$ is holomorphic for all $u^* \in \mathfrak{X}^*$," the entire theory presented above can be generalized. Furthermore, every equicontinuous semigroup creates a Γ for which it is a contraction semigroup (see [6]), and it turns out that the results given in Chapter IX of [10], along with several new theorems, can be obtained in this way. Details will appear in [7].

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