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THE ALGEBRA OF MULTIPLACE VECTOR-VALUED FUNCTIONS[1]

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1. In recent years there has been an upsurge of interest in the algebraic structure of systems of functions (operations, transformations, mappings) closed under various compositions (see, e.g., [1], [2], [4]). We have previously studied one-place functions, with one of our principal results being a complete axiomatic characterization of the semigroup of all partial transformations of a set [5]. In this note we report some of the results of our study of multiplace vector-valued functions. (An outline of a primitive stage of these studies appeared in [6]; a longer paper, with complete proofs and an extended bibliography will appear elsewhere.)

The nature and current state of development of the subject dictate a concrete approach. Accordingly, we work throughout with a given, arbitrary set S and with functions and operations defined in terms of elements of S. In such considerations, it soon becomes apparent that any restriction on the (finite) dimensionality or "fullness" of the domains and ranges of the functions is artificial and obstructive. Consequently, since there is no natural stopping-place, we go all the way, treating the totality of all multiplace vector-valued functions as a whole and admitting all (partial) transformations of Cartesian powers of S into Cartesian powers of S. The resulting system is delineated in §2; the naturalness of the structure may be judged from the results and applications mentioned in this and the two subsequent sections.

2. Let S be a nonempty set. For any positive integer n, let S^n denote the n-fold Cartesian power of S. The elements of S^n will usually be written as strings, e.g., $x_1x_2 \cdots x_n$. Let $\mathfrak{F}_{rd}$ denote the set of all functions whose domain is a nonempty subset of S^d and whose range

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is a subset of S^r; and let $\mathfrak{F}_\infty$ denote the union of all the sets $\mathfrak{F}_{rd}$, together with the empty function Φ. If $F\in\mathfrak{F}_{rd}$ then F has *degree* d, *rank* r, and *index* $d-r$, and we write $\delta F=d$, $\rho F=r$, $\iota F=d-r$. For any string $x_1\cdots x_{\delta F}$ in Dom F, the domain of F, the value of F at $x_1\cdots x_{\delta F}$ is denoted by $Fx_1\cdots x_{\delta F}$. Let $J_n\in\mathfrak{F}_{nn}$ denote the identity function on S^n. For any $F\in\mathfrak{F}_{rd}$, let RF denote the restriction of J_d to Dom F and LF the restriction of J_r to Ran F, the range of F; and set $R\Phi=L\Phi=\Phi$.

DEFINITION 1. Let F and G be functions in $\mathfrak{F}_\infty$. The serial composite FG of F and G is the function in $\mathfrak{F}_\infty$ specified as follows:

(a) If either $F=\Phi$ or $G=\Phi$, then $FG=\Phi$.

(b) If F and G are nonnull, then let $d=\max(\delta F+\iota G,\ \delta G)$ and set

$$\mathrm{Dom}(FG)=\{x_1\cdots x_d \mid x_1\cdots x_{\delta G}\in\mathrm{Dom}\,G,\ y_1\cdots y_{\delta F}\in\mathrm{Dom}\,F\}\tag{1}$$

where $x_1\cdots x_{\delta G}$ is the initial segment of length δG of the string $x_1\cdots x_d$, and $y_1\cdots y_{\delta F}$ is the initial segment of length δF of the string $(Gx_1\cdots x_{\delta G})\cdots x_d$. If Dom$(FG)$ is empty then $FG=\Phi$. Otherwise, FG is the function on Dom(FG) defined by:

$$(FG)x_1\cdots x_d=(Fy_1\cdots y_{\delta F})\cdots y_{d-\iota G},\tag{2}$$

for any string $x_1\cdots x_d\in\mathrm{Dom}(FG)$, where

$$y_1\cdots y_{d-\iota G}=(Gx_1\cdots x_{\delta G})\cdots x_d.\tag{3}$$

The basic idea behind serial composition is that output strings of one function, G, serve as input strings of another function, F, with the length of the original input strings (for G) adjusted to be the minimum necessary and sufficient to guarantee final output strings (from F) in all cases. Definition 1 combines three cases: $\delta F>\rho G$, $\delta F=\rho G$, $\delta F<\rho G$. In the case $\delta F=\rho G$, serial composition reduces to ordinary composition of (partial) transformations.

Serial composition is associative. Indeed, much more is true.

THEOREM 1. *Under the binary operation of serial composition and the two unary operations R and L, the set $\mathfrak{F}_\infty$ is a function semigroup with identity J_1* [5]. *This function semigroup has the right-subinverse property* [5].

Theorem 1 has many important consequences. It implies, first of all, that the entire system $\mathfrak{F}_\infty$ of multiplace, vector-valued functions under serial composition can be faithfully represented as a system of one-place functions under ordinary composition; and further, that the ideas and tools developed in [5] for the abstract study of one-place functions are relevant in this much more extensive setting.

THEOREM 2. *If FG is nonempty then* $\delta(FG)=\max(\delta F+\iota G, \delta G)$, $\rho(FG)=\max(\rho F, \rho G-\iota F)$ *and* $\iota(FG)=\iota F+\iota G$.

DEFINITION 2. Let F and G be functions in $\mathfrak{F}_\infty$. The parallel composite (F, G) of F and G is the function in $\mathfrak{F}_\infty$ specified as follows:

(a) If either $F=\Phi$ or $G=\Phi$, then $(F, G)=\Phi$.

(b) If F and G are nonnull, then let $m=\max(\delta F, \delta G)$ and set

$$\text{(4)}\quad \operatorname{Dom}(F, G)=\{x_1\cdots x_m \mid x_1\cdots x_{\delta F}\in\operatorname{Dom} F,\ x_1\cdots x_{\delta G}\in\operatorname{Dom} G\},$$

where $x_1\cdots x_{\delta F}$ and $x_1\cdots x_{\delta G}$ are the indicated initial segments of the string $x_1\cdots x_m$. If $\operatorname{Dom}(F, G)$ is empty then $(F, G)=\Phi$. Otherwise (F, G) is the function on $\operatorname{Dom}(F, G)$ defined by

$$\text{(5)}\quad \begin{aligned}(F, G)x_1\cdots x_m &= Fx_1\cdots x_{\delta F}Gx_1\cdots x_{\delta G},\\ &\qquad \text{for all } x_1\cdots x_m\in\operatorname{Dom}(F, G).\end{aligned}$$

THEOREM 3. *Parallel composition is associative, and is related to serial composition via the following conditional distributive and interassociative laws:*

$$\text{(6)}\quad (F, G)H = (FH, GH) \quad \textit{if and only if } \rho H \leqq \min(\delta F, \delta G);$$

$$\text{(7)}\quad F(G, H) = (FG, H) \quad \textit{if } \delta F \leqq \rho G.$$

THEOREM 4. *For any* $F, G\in\mathfrak{F}_\infty$, $R(F, G)=(RF)(RG)$; *and if* $FG\neq\Phi$, *then* $\delta(F, G)=\max(\delta F, \delta G)$, $\rho(F, G)=\rho F+\rho G$.

Let J and K denote the functions in $\mathfrak{F}_{12}$ defined as follows:

$$\operatorname{Dom} J = \operatorname{Dom} K = S^2,\quad Jx_1x_2 = x_2,\ Kx_1x_2 = x_1,\quad \text{for all } x_1x_2\in S^2.$$

Serial powers of K and J are defined by: $J^0=K^0=J_1$, $J^{n+1}=JJ^n$ *and* $K^{n+1}=KK^n$ *for* $n\geqq 0$.

THEOREM 5. *The elements of the semigroup generated by J and K under serial composition are the selectors, i.e., the functions that select a particular element from any string of a given length. Specifically, for* $1\leqq m\leqq n$, *we have:* $K^{n-m}J^{m-1}x_1\cdots x_n=x_m$ *for all* $x_1\cdots x_n\in S^n$.

The functions generated by J and K under *both* serial and parallel composition are called *multiselectors.* They are the functions that transform strings of a given length into other strings which are rearrangements, with possible repetitions and omissions, of the original strings. For example, each identity function J_n is a multiselector, as is the function in $\mathfrak{F}_{64}$ that transforms the string $x_1x_2x_3x_4$ into the string $x_2x_1x_1x_4x_1$. Since $K=J(J, J^0)$, we have

THEOREM 6. *The set of multiselectors is generated by J alone under serial and parallel composition.*

3. It appears that all the finitary compositions of multiplace functions and transformations that are found in the literature can be defined in a precise and variable-free manner within the system described above. To illustrate, we consider some of the compositions that occur most frequently. In each example, F will be an m-place function and $G_1, \cdots, G_m$ will have respective place-numbers $n_1, \cdots, n_m$.

A. Let $n_1= \cdots =n_m=n$. An n-place function A can be obtained via the $(m+1)$-ary operation conventionally indicated in terms of variables by:

$$A(x_1, \cdots, x_n) = F(G_1(x_1, \cdots, x_n), \cdots, G_m(x_1, \cdots, x_n)). \tag{8}$$

In the system described above, (8) can be expressed simply as

$$A = F(G_1, \cdots, G_m). \tag{9}$$

In (9) the parentheses and commas are operational rather than mere grouping symbols, and indicate how the $(m+1)$-ary operation is built up from binary operations: here $(m-1)$ iterated parallel compositions and one serial composition. It is also worth remarking that the often-noted "associative" property [**2**], [**4**] of this $(m+1)$-ary operation is simply a special case of the conditional right-distributive law (6).

B. Let $n_1+n_2+ \cdots +n_m=p$. A p-place function B can be obtained via the $(m+1)$-ary operation conventionally indicated by

$$\begin{aligned} B(x_1, \cdots, x_p) = F(G_1(x_1, \cdots, x_{n_1}), G_2(x_{n_1+1}, \cdots, x_{n_1+n_2}), \\ \cdots, G_m(x_{p-n_m+1}, \cdots, x_p)). \end{aligned} \tag{10}$$

Using J and its serial iterates, (10) can be expressed more succinctly in the form:

$$B = F(G_1, G_2J^{\delta G_1}, \cdots, G_mJ^{\delta G_1+\cdots+\delta G_{m-1}}). \tag{11}$$

The operation in (11) also has "associative" properties [**2**] which are simple consequences of (6).

C. In the literature, particularly in works on logic [**3**], one often encounters methods of generating functions that are equivalent to the employment of infinite sets of finitary operations. Each of these operations can be expressed in the form BM, where B is given by (11) and M is a multiselector of rank p. In particular, $B=BJ_p$, and A in (9) is given by $A=B(J_n, J_n, \cdots, J_n)$.

The presence of constant functions in $\mathfrak{F}_{11}$ makes a precise definition of such concepts as "place-fixing" and "essential variables" possible. For example, if F is a 2-place function, then to "fix F in its first place" means to replace F by $F(k, J_1)$, where k is some constant function in $\mathfrak{F}_{11}$; and the first place of F is "essential" if there are constant functions k_1, k_2 such that $F(k_1, J_1)\neq F(k_2, J_1)$. It is also easy to prove results such as the following: If F is an *associative* function [i.e., $F\in\mathfrak{F}_{12}$, Dom $F=S^2$, and $FF=F(J_1, FJ)$] and k_1, k_2 are constant functions, then

$$F(F(k_1, k_2), J_1) = F(k_1, J_1)F(k_2, J_1). \tag{12}$$

When the values of the constant functions k_1, k_2 are taken into account, then (12) is seen to be a purely function-theoretic restatement of the *left regular representation theorem*, the generalization to semigroups of the *Cayley theorem* for groups.

4. Let B be any binary operation on S, i.e., $B\in\mathfrak{F}_{12}$ and Dom $B=S^2$. Define functions $B_n\in\mathfrak{F}_{n,2n}$ and $\beta_n\in\mathfrak{F}_{n,n+1}$ by

$$B_nx_1\cdots x_ny_1\cdots y_n = Bx_1y_1Bx_2y_2\cdots Bx_ny_n, \qquad \text{for all } x_1\cdots x_ny_1\cdots y_n\in S^{2n},$$

$$\beta_nzx_1\cdots x_n = Bzx_1Bzx_2\cdots Bzx_n, \quad \text{for all } zx_1\cdots x_n\in S^{n+1}.$$

The function F in $\mathfrak{F}_\infty$ with Dom $F=S^{\delta F}$ is a *vector B-homomorphism* if

$$FB_Fx_1\cdots x_{\delta F}y_1\cdots y_{\delta F} = B_{\rho F}Fx_1\cdots x_{\delta F}Fy_1\cdots y_{\delta F};$$

and F is a *scalar B-homomorphism* if

$$F\beta_{\delta F}zx_1\cdots x_{\delta F} = \beta_{\rho F}zFx_1\cdots x_{\delta F}.$$

THEOREM 7. *If F and G are each vector (resp. scalar) B-homomorphisms, then both the serial composite FG and the parallel composite (F, G) are vector (resp. scalar) B-homomorphisms.*

Now consider the case where S is a field, with addition Σ and multiplication Π. Then the linear transformations of (finite-dimensional) vector spaces over S correspond precisely to those functions in $\mathfrak{F}_\infty$ which are simultaneously vector Σ-homomorphisms and scalar Π-homomorphisms. Thus, speaking informally, Theorem 7 says that both the serial composite and the parallel composite of linear transformations are linear. Since every linear transformation has a matrix representation, this yields two associative multiplications on the set of all finite matrices. Specifically, if M is an $m\times n$ matrix and N is a $p\times q$ matrix, then the serial product of M and N is the usual matrix product

$$M \times \left(\begin{array}{c|c} N & 0 \\ \hline 0 & I_{n-p} \end{array}\right),$$

when $n \geqq p$;

$$\left(\begin{array}{c|c} M & 0 \\ \hline 0 & I_{p-n} \end{array}\right) \times N,$$

when $n \leqq p$. Similarly, the parallel product of M and N is the matrix

$$\left(\begin{array}{c|c} \multicolumn{2}{c}{M} \\ \hline N & 0 \end{array}\right),$$

when $n \geqq q$;

$$\left(\begin{array}{c|c} M & 0 \\ \hline \multicolumn{2}{c}{N} \end{array}\right),$$

when $n \leqq q$.

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