EQUIVARIANT STABLE STEMS[1]

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Let $S^n(r)$ denote the n-sphere with a linear involution having a fixed point set of codimension r, where $0 \leqq r \leqq n$. We pick some fixed point as a base point and consider the set $[S^n(r); S^k(t)]$ of base point preserving equivariant homotopy classes of maps from $S^n(r)$ to $S^k(t)$. This has a natural group structure for $n-r \geqq 1$ and is abelian if $n-r \geqq 2$.

There is a suspension functor S without action and one Σ with action (that is, the reduced join with $S^1(0)$ and $S^1(1)$ respectively). These induce homomorphisms

$$[S^{n+1}(r); S^{k+1}(t)] \overset{S}{\longleftarrow} [S^n(r); S^k(t)] \overset{\Sigma}{\longrightarrow} [S^{n+1}(r+1); S^{k+1}(t+1)].$$

It can be shown that S is an epimorphism when $n \leqq 2k-1$ and $n-r \leqq 2(k-t)-1$ and is an isomorphism if the strict inequalities hold. Similarly, Σ is an epimorphism when $n \leqq 2k-1$ and $n-r \leqq k-1$ and is an isomorphism if the strict inequalities hold. By passing to the S-limit we define

$$\pi_n(r; t) = \lim_k [S^{n+k}(r); S^k(t)].$$

Σ induces $\Sigma\colon \pi_n(r; t) \to \pi_n(r+1; t+1)$ which is an epimorphism when $n \leqq r-1$ and an isomorphism when $n \leqq r-2$. By passing to the Σ-limit we define

$$\pi_{n,k} = \lim_t \pi_n(t+k; t).$$

There is the forgetful functor ψ (forgetting equivariance) and the fixed point set functor ϕ which yield homomorphisms

$$\pi_n \overset{\psi}{\longleftarrow} \pi_n(r; t) \overset{\phi}{\longrightarrow} \pi_{n-r+t} \tag{1}$$

where π_n denotes the classical n-stem. For the doubly stable groups these become

$$\pi_n \overset{\psi}{\longleftarrow} \pi_{n,k} \overset{\phi}{\longrightarrow} \pi_{n-k}. \tag{2}$$

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It is important to consider the generalization of these groups defined by

$$\pi_n(r, q; t) = \lim_k \left[S^{n+k}(r)/S^{n+k-r+q}(q); S^k(t)\right] \tag{3}$$

and we single out the case $q=0$ by the special notation

$$\overset{*}{\pi}_n(r; t) = \pi_n(r, 0; t).$$

If $r \geqq q \geqq p$ there is an exact sequence

$$\cdots \to \pi_n(r, q; t) \xrightarrow{\gamma} \pi_n(r, p; t) \xrightarrow{\rho} \pi_{n-r+q}(q, p; t) \xrightarrow{\delta} \pi_{n-1}(r, q; t) \to \cdots \tag{4}$$

and also a similar exact sequence with p deleted. A special case of interest is that for which $q=r-1$:

$$\cdots \to \pi_n \xrightarrow{\alpha} \pi_n(r, p; t) \xrightarrow{\beta} \pi_{n-1}(r-1, p; t) \xrightarrow{\psi} \pi_{n-1} \xrightarrow{\alpha} \pi_{n-1}(r, p; t) \to \cdots \tag{5}$$

(p may be deleted here). Also of interest is the case in which p is deleted and $q=0$:

$$\cdots \to \overset{*}{\pi}_n(r; t) \xrightarrow{i} \pi_n(r; t) \xrightarrow{\phi} \pi_{n-r+t} \xrightarrow{\Delta} \overset{*}{\pi}_{n-1}(r; t) \to \cdots. \tag{6}$$

(The maps ψ in (5) and ϕ in (6) are the forgetful and fixed point homomorphisms respectively.)

The Σ-suspension yields an *isomorphism*

$$\Sigma: \pi_n(r, q; t) \xrightarrow{\approx} \pi_n(r+1, q+1; t+1) \tag{7}$$

in all cases.

Let $\Phi(n)$ be the number of integers k with $0 < k \leqq n$ and $k \equiv 0, 1, 2$ or 4 (modulo 8). Our main general result is:

THEOREM. *If j is divisible by $2^{\Phi(r-q-1)}$ then there is an isomorphism*

$$\lambda_j: \pi_n(r, q; t) \xrightarrow{\approx} \pi_n(r, q; t+j).$$

If $2^{\Phi(r-q)} | j$ then λ_j commutes with ψ. In particular,

$$\overset{*}{\pi}_n(r; t) \approx \overset{*}{\pi}_n(r; t+j) \qquad \textit{for } 2^{\Phi(r-1)} | j$$

and

$$\overset{*}{\pi}_{n,k} \approx \overset{*}{\pi}_{n,k-j} \qquad \textit{for } 2^{\Phi(n+1)} | j.$$

Moreover, the period $2^{\Phi(n+1)}$ for $\pi^*_{n,k}$ is best possible when $n+1 \equiv 0, 1, 2, 4$ (modulo 8).

Using this result, (7), Spanier-Whitehead duality and results of Atiyah [1] and James [2] it is easy to show that there is an isomorphism

$$\pi^{n}_{k,r} \approx \pi^{*}_{n}(r; r+k) \quad \text{when } n < k-1. \tag{8}$$

Here $\pi^n_{k,r} = \pi_{k+n}(V_{k+r,r})$ where $V_{k+r,r}$ is the Stiefel manifold $O(k+r)/O(k)$. In particular, if $n<k-1$ and $n<r-1$, $\pi^n_{k,r} \approx \pi^*_{n,-k}$. The periodicity, in k, of the $\pi^n_{k,r}$ which results from (8) is known and is due to James [2]. Also see [3].

The groups $\pi^n_{k,r}$ have been calculated by Paechter [4] for $n \leqq 5$. Our methods, which are aided by the relationships between $\pi^*_n(r; r+k)$ and $\pi_n(r; r+k)$, allow us to calculate the groups $\pi^*_n(r; r+k)$ for $n \leqq 8$ (with a few ambiguities) and their orders for $n \leqq 10$.

Also the homomorphisms ψ and ϕ are determined in roughly this range. We shall comment here only on ϕ. First, there is the general result:

PROPOSITION. *$\phi: \pi_{n,k} \to \pi_{n-k}$ is onto if $n \geqq 2k$. It is also onto, and splits, if $k \leqq 0$. If $n<0$, ϕ is an isomorphism.*

The calculations yield the following special results:

$$\phi: \pi_{n,n} \overset{\approx}{\to} 2^n \pi_0 \quad \text{for } 1 \leqq n \leqq 4,$$

$$\phi: \pi_{5,5} \overset{\approx}{\to} 16\pi_0,$$

$$\operatorname{Im}\{\phi: \pi_{n,n} \to \pi_0\} \subset 16\pi_0 \quad \text{for } n \geqq 4.$$

There are also exact sequences:

$$0 \to Z_2 \to \pi_{6,6} \overset{\phi}{\to} 16\pi_0 \to 0,$$

$$0 \to Z_2 \oplus Z_2 \to \pi_{7,7} \overset{\phi}{\to} 16\pi_0 \to 0.$$

$$0 \to Z_{16} \to \pi_{8,8} \overset{\phi}{\to} 16\pi_0 \to Z_2.$$

For $n-k=1, 2, 3$ we obtain

$$\phi: \pi_{n,n-1} \to \pi_1 \text{ is } \begin{cases} \text{onto for } n \leqq 3, \\ \text{zero for } n \geqq 4; \end{cases}$$

$$\phi: \pi_{n,n-2} \to \pi_2 \text{ is } \begin{cases} \text{onto for } n \leqq 6, \\ \text{zero for } n \geqq 7, \end{cases}$$

$$\phi\colon \pi_{n,n-3} \to \pi_3 \text{ is } \begin{cases} \text{onto } \pi_3 \text{ for } n \leqq 7, \\ \text{onto } 2\pi_3 \text{ for } n = 8, \\ \text{onto } 4\pi_3 \text{ for } n = 9, \\ \text{onto } 8\pi_3 \text{ for } n \geqq 10. \end{cases}$$

The kernel of ϕ can be calculated in the range $n \leqq 8$. (As far as we are aware the only previously known result in this direction was that $\phi\colon \pi_{n,n} \to \pi_0$ has image contained in $2\pi_0$ for $n \geqq 1$; known since it is an easy consequence of Smith theory.)

The calculations of the $\pi_{n,k}$ are accomplished mainly through the use of spectral sequences associated with the exact sequence (4).

The only difficulties in calculating the $\pi_{n,k}$ are in dealing with the 2-primary components. In fact, if $\mathcal{C}$ denotes the class of finite 2-groups we can prove that

$$\phi\colon \pi_{n,k} \overset{\approx}{\to} \pi_{n-k} \bmod \mathcal{C} \text{ if } k \text{ is odd,}$$

$$\phi \oplus \psi\colon \pi_{n,k} \overset{\approx}{\to} \pi_{n-k} \oplus \pi_n \bmod \mathcal{C} \text{ if } k \text{ is even.}$$

An interesting corollary of the proof of the periodicity theorem is worth mentioning here. Let T_k be the matrix

$$\begin{pmatrix} -I_k & 0 \\ 0 & I \end{pmatrix} \in O = \bigcup_{m=1}^{\infty} O(m).$$

Consider the antipodal involution A on S^n and the left translation by T_k on O. Then we can show that an equivariant map

$$(S^n, A) \to (O, T_k)$$

exists if and only if $2^{\Phi(n)} \mid k$. The "if" part is easy and, in fact, when $2^{\Phi(n)} \mid k$ we construct an equivariant map

$$(S^n, A) \to (O(k), -I).$$

The proofs of these results will be published elsewhere.

Note added in proof. It has been brought to our attention that the groups $\pi_{k,r}^n$ have been calculated for $n \leqq 13$ and r large by C. S. Hoo and M. E. Mahowald. Their results are tabulated in Bull. Amer. Math Soc. **71** (1965), 661–667. Unfortunately, it does not appear that their methods could give any information on the fixed point homomorphism. However, a comparison of their results with our methods does strongly indicate the *conjecture* that the image of $\phi\colon \pi_{n,n} \to \pi_0$ is $a_n\pi_0$ where $a_8 = a_9 = 2^5$, $a_{10} = 2^6$, $a_{11} = 2^7$, $a_{12} = a_{13} = a_{14} = 2^8$. (For $n \leqq 7$, a_n is given above.)

References

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