SEMI-IDEMPOTENT MEASURES ON ABELIAN GROUPS[1]

BY IRVING KESSLER

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Let $M(G)$ denote the set of complex valued regular Borel measures on a compact abelian group G. We assume that Γ, the dual group of G, is a totally ordered group. Let $F(G)$ denote the set of all $\mu \in M(G)$ such that the Fourier transform $\hat{\mu}$ of μ is an integer valued function. A measure μ is idempotent if $\hat{\mu}$ assumes only the values 0 or 1. A $\mu \in M(G)$ is semi-idempotent if $\hat{\mu}(\gamma)=0$ or 1 for all $\gamma>0$ in Γ.

The purpose of this note is to sketch a proof of the following theorem.

THEOREM 1. *If $\mu \in M(G)$ and $\hat{\mu}(\gamma)$ is an integer for all $\gamma>0$ in Γ, then there exists a $\lambda \in F(G)$ such that $\hat{\lambda}(\gamma)=\hat{\mu}(\gamma)$ for all $\gamma>0$. In particular, if μ is a semi-idempotent measure on G, then there exists an idempotent measure λ on G such that $\hat{\mu}(\gamma)=\hat{\lambda}(\gamma)$ for all $\gamma>0$ in Γ.*

This result was obtained by Helson [**3**], for the case $G=T$, the circle group, $\Gamma=Z$ the integer group. Also a special case of Theorem 1 for $G=T^2$, $\Gamma=Z^2$ was proven by Rudin [**6**].

OUTLINE OF PROOF. We assume first that $G=T^k$, the k-dimensional torus group and $\Gamma=Z^k$ is a totally ordered group. Then $Z^k=\Gamma_1\oplus\Gamma_2$ where Γ_2 is a subgroup of the reals, Γ_1 is a totally ordered group, and $\Gamma_1\oplus\Gamma_2$ is lexicographically ordered from the right. Let $\mu \in M(G)$ such that $\hat{\mu}(\gamma)$ is an integer for all $\gamma>0$ in Z^k.

Let $E(\mu)=\{\gamma \in Z^k: \gamma>0 \text{ and } \hat{\mu}(\gamma)\neq 0\}$ and, for every positive integer n, let

$$A_n = \{\gamma\cdot\mu \in M(G): \gamma = (\gamma_1, \gamma_2) \in E(\mu) \text{ and } \gamma_2 > n\}.$$

We then prove the following

LEMMA. *If $E(\mu)$ and A_n are defined as above, we have either*

(1) $E(\mu)$ is contained in a finite union of $(k-1)$-dimensional hyperplanes, or

(2) $A_n \neq \varnothing$ for every n.

If (1), then Theorem 1 follows by induction. If (2), we set

[1] This is an announcement of a portion of the author's dissertation at the University of Wisconsin written under the direction of Professor Walter Rudin.

$$A = \bigcap_{n=1}^{\infty} \overline{A}_n$$

($\overline{A}_n$ is the weak*-closure of A_n). Then A is nonempty and weak*-compact and hence contains an element $\nu \neq 0$, of minimal norm. Then $\nu \in F(G)$, and applying a lemma of Ito and Amemiya [**5**], we find that ν is a measure of the form $\gamma \cdot \chi_H \mu$ where χ_H is the characteristic function of a compact subgroup H of G, and $\gamma \in \Gamma$.

Let $\mu_1 = \chi_H \mu$. Then $\mu = \mu_1 + (\mu - \mu_1)$ is an orthogonal decomposition of μ where $\mu_1 \in F(G)$ and $(\hat{\mu} - \hat{\mu}_1)(\gamma)$ is an integer for all $\gamma > 0$ in Γ. Now we can apply the argument above to the measure $\mu - \mu_1$. Since the norm of $\mu - \mu_1$ decreases at least one from that of μ, we see that after a finite number of steps, we obtain

$$\mu = \sum_{i=1}^{n} \mu_i + \tau$$

where $\mu_i \in F(G)$ for $i = 1, \cdots, n$, and $\hat{\tau}(\gamma) = 0$ for all $\gamma > 0$.

The proof of the theorem for arbitrary compact G is completed by a transfinite induction on the cardinality of Γ.

In the proof of the lemma, we used the following theorem of P. J. Cohen [**1**] and Davenport [**2**].

THEOREM. *Let Γ be a totally ordered group. Let $E = \{\gamma_1, \gamma_2, \cdots, \gamma_N\}$ be a fixed set of N positive elements in Γ, $N \geqq 3$. Suppose $\mu \in M(G)$, and*

$$|\mu(\gamma)| \geqq 1 \qquad \textit{for } \gamma \in E,$$
$$\mu(\gamma) = 0 \qquad \textit{for } \gamma > 0, \gamma \notin E.$$

Then there exists a constant k, independent of the group G, such that

$$\|\mu\| > k(\log N / \log \log N)^{1/4}.$$

This theorem was proven by Cohen [**1**] and Davenport [**2**]. An examination of the proofs in these papers shows that they actually obtain the above theorem although they only considered the case where $\mu(\gamma) = 0$ for *all* $\gamma \notin E$. See also Hewitt and Zuckerman [**4**] for the case in which the torsion subgroup of Γ is an arbitrary finite group.

We also have proven

THEOREM 2. *Let $Q = \{(n_1, \cdots, n_k) \in Z^k: n_i \geqq 0$ for $i = 1, \cdots, k\}$. Suppose $\mu \in M(T^k)$ and $\hat{\mu}(q)$ is an integer for all $q \in Q$. Then there exists a $\lambda \in F(T^k)$ such that $\hat{\mu}(q) = \hat{\lambda}(q)$ for all $q \in Q$.*

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UNIVERSITY OF WISCONSIN AND UNIVERSITY OF MICHIGAN