# HIGHER RANK CLASS GROUPS[1]

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Let $A$ be a noetherian ring which is locally Macaulay. For each integer $i \geqq 0$, groups $C_i(A)$ and $W_i(A)$ are defined, each sequence of groups generalizing to higher dimensions the usual class group of an integrally closed noetherian domain. $C_i(A)$ is called the $i$th *class group* of $A$, and $W_i(A)$ is called the $i$th *homological class group* of $A$. The main purpose of this note is to show that both sequences of groups have properties analogous to the class group of a Noetherian integrally closed integral domain, and finally to establish a connection between them.

1. Throughout this section $A$ is a commutative noetherian ring which is locally Macaulay. A set of elements $x_1, \cdots, x_s$ is an $A$-*sequence* of length $s$ if $x_1A + \cdots + x_sA \neq A$ and $x_1A + \cdots + x_iA : x_{i+1} = x_1A + \cdots + x_iA$ for $i = 0, 1, \cdots, s-1$. Count the empty set as an $A$-sequence of length 0 and specify that it generate the zero ideal of $A$.

Note that if $x_1, \cdots, x_s$ is an $A$-sequence of length $s$, then $x_1A + \cdots + x_sA$ is an unmixed ideal of $A$ of height $s$.

For each $i \geqq 0$, form the free abelian group on the generators $\langle \mathfrak{p} \rangle$ where $\mathfrak{p}$ is a height $i$ prime ideal of $A$. This group will be denoted by $D_i(A)$. For each $A$-sequence $x_1, \cdots, x_i$, consider the element $\sum e(x_1, \cdots, x_i | A_{\mathfrak{p}}) \langle \mathfrak{p} \rangle$ of $D_i$ (here $e(y_1, \cdots, y_i | M)$ denotes the multiplicity of $y_1A + \cdots + y_iA$ on $M$). Let $R_i$ designate the subgroup of $D_i$ generated by all such elements. Set $C_i(A) = D_i(A)/R_i$ and call $C_i(A)$ the *class group of rank i* for $A$. Denote the image of $\langle \mathfrak{p} \rangle$ in $C_i(A)$ by $\mathrm{cl}(\mathfrak{p})$. Set $C.(A) = \oplus C_i(A)$.

EXAMPLES. $C_0(A)$ is always finitely generated. $C_0(A)$ is finite if and only if (0) is a primary ideal of $A . C_0(A) = 0$ if and only if $A$ is a domain.

If $A$ is a Dedekind domain, then $C_1(A)$ is the ordinary ideal class group of $A$. More generally, if $A$ is integrally closed, then $C_1(A)$ is the class group of $A$ [1, §1, no. 10].

We have not been able to locate the following lemma in the literature.

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LEMMA 1.1. *Let $S$ be a multiplicatively closed subset of $A$. If $y_1, \cdots, y_i$ is an $A_S$-sequence, then there is an $A$-sequence $x_1, \cdots, x_i$ such that $\sum y_i A_S = \sum x_i A_S$.*

THEOREM 1.2. (*Cf.* [1, *Proposition* 17, §1, *no.* 10].) *Let $S$ be a multiplicatively closed subset of $A$. Then for each $i \geqq 0$, there is an epimorphism $C_i(A) \to C_i(A_S)$ deduced from $\langle \mathfrak{p} \rangle \to 0$ if $\mathfrak{p} \cap S \neq \emptyset$ and $\langle \mathfrak{p} \rangle \to \langle \mathfrak{p} A_S \rangle$ if $\mathfrak{p} \cap S = \emptyset$. The kernel is generated by $\{\mathrm{cl}(\mathfrak{p})\}$ where $ht(\mathfrak{p}) = i$ and $\mathfrak{p} \cap S \neq \emptyset$.*

COROLLARY 1.3. (*Cf.* [4, *Lemma* 1.7].) *If $\mathfrak{p} \cap S \neq \emptyset$ implies that $\mathrm{cl}(\mathfrak{p}) = 0$, then the epimorphism of Theorem* 1.2 *is an isomorphism.*

COROLLARY 1.4. *If $C_i(A_S) = 0$, then $C_i(A)$ is generated by $\{\mathrm{cl}(\mathfrak{p})\}$ where $ht(\mathfrak{p}) = i$ and $\mathfrak{p} \cap S \neq \emptyset$.*

COROLLARY 1.5. *There is an epimorphism $C_i(A) \to \bigoplus_{ht(\mathfrak{p})=i} C_i(A_{\mathfrak{p}})$ deduced from $\langle \mathfrak{p} \rangle \to \langle \mathfrak{p} A_{\mathfrak{p}} \rangle$.*

THEOREM 1.6. *If $x_1, \cdots, x_k$ is an $A$-sequence, then there is a homomorphism $C_i(A/\sum x_s A) \to C_{i+k}(A)$ whose image is the subgroup of $C_{i+k}(A)$ generated by $\{\mathrm{cl}(\mathfrak{p})\}$ where $ht(\mathfrak{p}) = i+k$ and $\mathfrak{p} \supseteq \sum x_s A$.*

With Theorem 1.2, this yields

COROLLARY 1.7. *Suppose that $x$ is an $A$-sequence. Then the sequence*

$$C_i(A/xA) \to C_{i+1}(A) \to C_{i+1}(A[x^{-1}]) \to 0$$

*is exact.*

An application of the associative law for multiplicities yields

THEOREM 1.8. *If $ht(\mathfrak{p}) = k$ and $\mathrm{cl}(\mathfrak{p}) = 0$, then there is a homomorphism $C_i(A/\mathfrak{p}) \to C_{i+k}(A)$ whose image is the subgroup of $C_{i+k}(A)$ generated by the* $\mathrm{cl}(\mathfrak{q})$ where *$ht\ (\mathfrak{q}) = i+k$ and $\mathfrak{q} \supseteq \mathfrak{p}$.*

Using techniques similar to those of [2, Proof of Proposition 7-1] we get

LEMMA 1.9. *Suppose that $C_i(A_{\mathfrak{p}}) = 0$ for each prime ideal $\mathfrak{p}$ of height $i$ of $A$. Then $C_{i+1}(A[X])$ is generated by $\{\mathrm{cl}(\mathfrak{q} A[X])\}$ where $\mathfrak{q}$ ranges over the prime ideals of $A$ of height $i+1$.*

THEOREM 1.10. *If $C_i(A_{\mathfrak{p}}) = 0$ for all prime ideals $\mathfrak{p}$ of $A$ of height $i$, then there is an epimorphism $C_{i+1}(A) \to C_{i+1}(A[X])$.*

COROLLARY 1.11. (Cf. [1, *Corollary to Theorem* 2].) *$C.(A) = 0$ implies $C.(A[X]) = 0$.*

REMARK. Corollary 1.11 does not hold for power series adjunction as Samuel's example [4] shows.

COROLLARY 1.12. *If $F$ is a field, then* $C.(F[X_1, \cdots, X_n])=0$.

COROLLARY 1.13. *Let the Krull dimension of $A$ be $n<\infty$. Suppose that $C_n(A_{\mathfrak{p}})=0$ for each prime ideal $\mathfrak{p}$ of $A$ of height $n$. Then $C_{n+1}(A[X])=0$.*

A theorem similar to Theorem **1.10** is

THEOREM 1.14. *Let $A$ and $B$ be finitely generated over a field $F$. Suppose, that for each $i\geqq 0$, $C_i(A_{\mathfrak{p}})=0$ for any prime ideal $\mathfrak{p}$ of height $i$ of $A$, and that $C.(K\otimes_F B)=0$ for any overfield $K$ of $F$. Then there is an epimorphism $C_j(A)\rightarrow C_j(A\otimes_F B)$ given by $\mathrm{cl}(\mathfrak{p})\rightarrow \mathrm{cl}(\mathfrak{p}\otimes_F B)$. In particular, $C.(A)=0$ implies $C.(A\otimes_F B)=0$.*

THEOREM 1.15. *For $i\geqq 1$, $C_i(A_1\oplus A_2)=C_i(A_1)\oplus C_i(A_2)$.*

2. Let $A$ be a commutative noetherian ring. The hypotheses on $A$ in §1 need not be assumed in order to define the groups $W_i(A)$. The reader is referred to [3] for the $K$-theory needed here.

Let $\mathfrak{M}_i(A)=\mathfrak{M}_i$ denote the category of finitely generated $A$-modules $M$ such that $M_{\mathfrak{p}}=0$ for all prime ideals $\mathfrak{p}$ of $A$ with $ht(\mathfrak{p})<i$. Then $\mathfrak{M}_j$ is a Serre subcategory of $\mathfrak{M}_i$ for all $j>i$. Let $K^i(\mathcal{C})$ denote the $i$th Grothendieck group of the category $\mathcal{C}$ for $i=0$, 1. If $C\in\mathcal{C}$, then $\gamma(C)$ denotes the image (or class) of $C$ in $K^0(\mathcal{C})$.

PROPOSITION 2.1. *$K^0(\mathfrak{M}_i/\mathfrak{M}_{i+1})$ is isomorphic to $D_i(A)$, the isomorphism being given by the length function.*

Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & K^0(\mathfrak{M}_{i+2}) & \xrightarrow{=} & K^0(\mathfrak{M}_{i+2}) & & \\
 & & \downarrow & & \downarrow & & \\
K^1(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \rightarrow & K^0(\mathfrak{M}_{i+1}) & \xrightarrow{f} & K^0(\mathfrak{M}_i) & \xrightarrow{g} & D_i(A) \rightarrow 0 \\
=\downarrow & & \downarrow & & \downarrow & & \downarrow = \\
K^1(\mathfrak{M}_i/\mathfrak{M}_{i+1}) & \rightarrow & D_{i+1}(A) & \xrightarrow{f'} & K^0(\mathfrak{M}_i/\mathfrak{M}_{i+2}) & \xrightarrow{g'} & D_i(A) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Because each element in $\mathfrak{M}_i/\mathfrak{M}_{i+1}$ has finite length, each of the rows is exact. The columns are also exact.

Since the group $D_i(A)$ is free, the kernels of $g$ and $g'$ in the above diagram are direct summands of their respective domains. For each

$i \geqq 0$ define the group $Z_{i+1}(A)$, and the *homological class group of rank* $i+1$, $W_{i+1}(A)$, to be the kernels of $g$ and $g'$ respectively. Since the rows are exact this is the same as saying that $Z_{i+1}(A)$ is the image of $f$ and $W_{i+1}(A)$ is the image of $f'$. Moreover

$$K^0(\mathfrak{M}_i) = Z_{i+1}(A) \oplus D_i(A)$$

and

$$K^0(\mathfrak{M}_i/\mathfrak{M}_{i+2}) = W_{i+1}(A) \oplus D_i(A).$$

The results of [3] yield

PROPOSITION 2.2. *$K^1$ $(\mathfrak{M}_i/\mathfrak{M}_{i+1})$ is isomorphic to the direct sum of the groups of units of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, $ht(\mathfrak{p})=i$. Consequently the kernel of $f'$ is generated by the $\gamma(A/(\mathfrak{p}+xA))$, $x\notin\mathfrak{p}$, as $\mathfrak{p}$ ranges over the prime ideals of $A$ of height $i$, and hence $W_{i+1}(A)$ is $D_{i+1}(A)$ modulo the subgroup generated by these.*

By convention $W_0(A)=0$. Set $W^\bullet(A)=\oplus W_i(A)$.
Diagram chasing will give

THEOREM 2.3. *Let $S$ be a multiplicatively closed subset of $A$. For each $i$ there is an epimorphism $W_i(A)\to W_i(A_S)$ induced by the functor $A_S\otimes_A -$. The kernel is generated by $\gamma(A/\mathfrak{p})$ with $\mathfrak{p}\cap S\neq\varnothing$, $ht(\mathfrak{p})=i$.*

COROLLARY 2.4. *If for each prime ideal $\mathfrak{p}$ of $A$ of height $i$ with $\mathfrak{p}\cap S\neq\varnothing$, $\gamma(A/\mathfrak{p})=0$ in $K^0(\mathfrak{M}_{i-1}/\mathfrak{M}_{i+1})$ then the epimorphism of Theorem 2.3 is an isomorphism.*

COROLLARY 2.5. *If $W_i(A_S)=0$, then $W_i(A)$ is generated by $\{\gamma(A/\mathfrak{p})\}$, $ht(\mathfrak{p})=i$, $\mathfrak{p}\cap S\neq\varnothing$.*

COROLLARY 2.6. *The functors $A_\mathfrak{p}\otimes_A -$ induce an epimorphism $W_i(A)\to\oplus_{ht(\mathfrak{p})=i} W_i(A_\mathfrak{p})$.*

THEOREM 2.7. *Let $A$ be locally Macaulay, $I$ an unmixed ideal of height $k$. Then there is a homomorphism*

$$W_i(A/I) \to W_{i+k}(A)$$

*induced by considering each $A/I$-module as an $A$-module. The image is generated by the $\gamma(A/\mathfrak{p})$, $\mathfrak{p}$ a prime ideal of height $i+k$ containing $I$.*

Using Theorems 2.3 and 2.7 one gets

THEOREM 2.8. *Let $x$ be an $A$-sequence, $A$ a locally Macaulay ring. Then the sequence*

$$W_i(A/xA) \to W_{i+1}(A) \to W_{i+1}(A[x^{-1}]) \to 0$$

*is exact.*

THEOREM 2.8. *The functor* $A[X]\otimes_A -$ *induces an epimorphism* $W_i(A) \to W_i(A[X])$. *Furthermore* $W_{n+1}(A[X])=0$ *if the Krull dimension of* $A$ *is* $n<\infty$.

COROLLARY 2.9. $W^\bullet(A)=0$ *implies* $W^\bullet(A[X])=0$.

COROLLARY 2.10. $W^\bullet(F[X_1, \cdots, X_n])=0$ *when* $F$ *is a field.*

THEOREM 2.11. *Let* $A_1$ *and* $A_2$ *be two rings. Then*

$$W_i(A_1 \oplus A_2) = W_i(A_1) \oplus W_i(A_2).$$

3. It is natural to ask if $C_i(A)=W_i(A)$ when both are defined. There are several results in this direction.

THEOREM 3.1. $W_i(A)$ *is a homomorphic image of* $C_i(A)$.

THEOREM 3.2. *If* $C_i(A)=0$, *then* $W_{i+1}(A)=C_{i+1}(A)$.

COROLLARY 3.3. *If* $A$ *is a domain, then* $W_1(A)=C_1(A)$.

COROLLARY 3.4. $C^\bullet(A)=0$ *if, and only if,* $A$ *is an integral domain and* $W^\bullet(A)=0$.

For an example which shows that in general $W^\bullet(A)\neq C^\bullet(A)$ let $Q$ be a primary ring which is not a field and set $A=Q[X]$. Then $W_1(A)=0$ while $C_1(A)$ is an infinite group.

## REFERENCES

1. N. Bourbaki, *Algebre commutative*, Chapitre 7, Hermann, Paris, 1965.
2. L. Claborn, *On the theory of E-rings*, Doctoral Dissertation, University of Michigan, Ann Arbor, Michigan, 1963.
3. A. Heller, *Some exact sequences in algebraic K-theory*, Topology 3 (1965), 389–409.
4. P. Samuel, *On unique factorization domains*, Illinois J. Math. 5 (1961), 1–17.

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