LOCALLY COMPACT TRANSFORMATION GROUPS AND C^*-ALGEBRAS

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It has long been recognized that one may associate operator algebras with transformation groups (see, e.g. [9, Chapter III], [11; 1, p. 310] [5]). In this paper we shall answer two questions about the ergodic invariant probability measures on a locally compact transformation group (Theorems 2 and 3). This information is then used to solve analogous problems for the unit traces of a C^*-algebra (Theorems 5 and 6). Full proofs will appear elsewhere.

Let (G, Z) be a topological transformation group with G and Z second countable and Hausdorff, G a locally compact group, and Z a compact space. Let Z/G be the set of orbits $G\zeta$ with ζ in Z, together with the quotient topology. Define an equivalence relation $\sim$ on Z/G by $p\sim q$ if the sets $\{p\}$ and $\{q\}$ have the same closure, and let $(Z/G)^\sim$ be the equivalence classes with the quotient topology (see [8, p. 58]). The elements of $(Z/G)^\sim$ are in one-to-one correspondence with the subsets of Z that are closures of orbits. Z/G is T_0 if and only if $\sim$ is trivial, and T_1 if and only if the orbits are closed.

Let Z be compact and let $C(Z)$ be the continuous complex valued functions on Z with the uniform norm, and $M(Z)=C_r(Z)^*$ the real Radon measures on Z with the weak* topology. Let G act on $C(Z)$ and $M(Z)$ by translation, i.e., for s in G, ζ in Z, f in $C(Z)$, and μ in $M(Z)$, let

$$(sf)(\zeta) = f(s^{-1}\zeta),$$
$$(s\mu)(f) = \mu(s^{-1}f).$$

Let $M_G(Z)$ be the invariant measures on Z, and $P_G(Z)$ the corresponding probability measures, i.e.,

$$P_G(Z) = M_G^+(Z) \cap H,$$

where $M_G^+(Z)$ are the positive invariant measures, and H are the measures μ such that $\mu(Z)=1$. $P_G(Z)$ is a compact simplex in the sense of Choquet, and its extremal points are just the ergodic mea-

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sures (see [**10**, §10]). Let $EP_G(Z)$ be the extreme points together with the simplex structure topology. A subset is defined to be closed in this topology if it consists of the extreme points of a closed face in $P_G(Z)$. $EP_G(Z)$ is compact, and it is Hausdorff if and only if $EP_G(Z)$ is closed in $P_G(Z)$ (see [**4**]).

The support of an ergodic measure is the closure of an orbit (see [**8**, p. 59]), hence we may define a map

$$\theta: EP_G(Z) \to (Z/G)^\sim$$

by letting $\theta(\mu)$ correspond to the support of μ. The following is verified:

THEOREM 1. *θ is continuous. In addition*

(a) *If Z/G is T_0, then θ is one-to-one.*

(b) *If the orbits are closed, θ is onto.*

(c) *If the orbits are finite, and uniformly bounded in cardinality, θ is a homeomorphism.*

(d) *If the orbits have the same finite cardinality, then $EP_G(Z)$ and $(Z/G)^\sim$ are Hausdorff.*

(e) *If G is equicontinuous, θ is a homeomorphism onto, and $EP_G(Z)$ and $(Z/G)^\sim$ are Hausdorff.*

In particular, if all of the orbits are finite, then θ is a continuous bijection. On the other hand our first construction shows:

THEOREM 2. *There is a distal action of the integers G on a compact metric space Z such that all of the orbits are finite, but θ is not a homeomorphism.*

For many transformation groups, θ is not one-to-one. In fact we have proved:

THEOREM 3. *There is a C^∞ distal action of the integers G on the torus Z such that $(Z/G)^\sim$ has only one point (i.e., (G, Z) is minimal), and $EP_G(Z)$ is uncountable.*

Let $\mathfrak{A}$ be a separable C^*-algebra with identity. Let pr $\mathfrak{A}$ be the set of primitive ideals in $\mathfrak{A}$ with the Jacobson structure topology (see [**3**, §3]). Let $\mathfrak{A}^*$ be the Banach dual of $\mathfrak{A}$ with the weak* topology. The central functions $C(\mathfrak{A})$ are the f in $\mathfrak{A}^*$ such that $f(AB)=f(BA)$ for all A and B in $\mathfrak{A}$. Let $T(\mathfrak{A})$ be the unit traces on $\mathfrak{A}$, i.e.,

$$T(\mathfrak{A}) = C^+(\mathfrak{A}) \cap H,$$

where $C^+(\mathfrak{A})$ are the positive central functions, and H consists of the f in $\mathfrak{A}^*$ such that $f(I)=1$. $T(\mathfrak{A})$ is a compact simplex (see [**12**, Satz **1**]), and its extreme points are just the traces that give rise to factor representations (see [**3**, §6.7.3]). Let $ET(\mathfrak{A})$ be the extreme traces, with the simplex structure topology.

The kernel of a factor representation of $\mathfrak{A}$ is primitive (see [**2**, p. 100]). This enables us to define a map

$$\theta' : ET(\mathfrak{A}) \to \mathrm{pr}\ \mathfrak{A}$$

by

$$\theta'(\tau) = \text{kernel } L^{\tau},$$

where L^{τ} is the representation defined by τ.

THEOREM. 4. *θ' is continuous. In addition,*

(a) *If $\mathfrak{A}$ is of type I, then θ' is one-to-one.*
(b) *If all of the representations of $\mathfrak{A}$ are finite dimensional, then θ' is onto.*
(c) *If the irreducible representations of $\mathfrak{A}$ have dimension uniformly bounded by a finite cardinal, then θ' is a homeomorphism.*
(d) *If all of the irreducible representations of $\mathfrak{A}$ are of the same finite dimension, then $ET(\mathfrak{A})$ and* pr $\mathfrak{A}$ *are Hausdorff.*

Letting $\mathfrak{A}(G, Z)$ be the C^*-algebra associated with a transformation group (G, Z) (see [**6**, p. 890]) we may use Theorems 2 and 3 to prove

THEOREM 5. *There is a separable C^*-algebra $\mathfrak{A}$ for which all of the representations are finite dimensional, and θ' is not a homeomorphism.*

THEOREM 6. *There is a separable C^*-algebra $\mathfrak{A}$ such that* pr $\mathfrak{A}$ *has only one point (i.e., $\mathfrak{A}$ is simple), and $ET(\mathfrak{A})$ is uncountable.*

Sketching the proofs of Theorems 5 and 6, *assume that G is discrete and Z is compact*, and let $\mathfrak{A} = \mathfrak{A}(G, Z)$. Consider the diagram

$$\begin{array}{ccc} EP_G(Z) & \overset{\pi'}{\longleftarrow} & ET(\mathfrak{A}) \\ \downarrow \theta & & \downarrow \theta' \\ (Z/G)^{\sim} & \overset{\pi}{\longleftarrow} & \mathrm{pr}\ \mathfrak{A} \end{array}$$

$$\overrightarrow{T}$$

θ and θ' are defined above. $C(Z)$ may be regarded as a subalgebra of $\mathfrak{A}$, and if P is a primitive ideal in $\mathfrak{A}$, there is an orbit closure F in Z such that

$$P \cap C(Z) = \{f \in C(Z) : f \mid F = 0\}.$$

$\pi(P)$ is defined to be the corresponding element of $(Z/G)^{\sim}$. π is continuous and onto.

If τ is an extremal trace, its restriction to $C(Z)$ is an ergodic measure (see [**12**, Lemma 14]). Letting π' be the restriction map, π' is continuous and onto (see [**12**, Lemma 16]), and the diagram is commutative.

The isotropy group H_ζ at ζ consists of the s in G for which $s\zeta=\zeta$. Irreducible representations of H_ζ may be induced to irreducible representations of $\mathfrak{A}$ (see [**6**, p. 901]). Inducing the trivial one-dimensional representation at ζ, one obtains a map T_1 of Z into pr $\mathfrak{A}$. If the isotropy groups "vary continuously" with ζ, it follows from [**6**, Theorem 2.1] that T_1 defines a continuous map T of $(Z/G)^{\sim}$ into pr $\mathfrak{A}$ which is a cross-section for π. This condition on isotropy groups is too strong for our purposes. We have been able to prove:

THEOREM 7. *If the isotropy groups are commutative, then T_1 induces a continuous cross-section T for π.*

We have also generalized Theorem 7 to locally compact G and Z.

Turning to Theorem 5, let $\mathfrak{A}=\mathfrak{A}(G, Z)$, where (G, Z) is described in Theorem 2. As the orbits are closed, the action of G on Z is smooth, and by Mackey's Imprimitivity Theorem, all of the irreducible representations of $\mathfrak{A}$ are induced from characters on isotropy groups H_ζ (see [**6**, Theorem 2.2]). As the latter are of finite index in G, the irreducible representations are finite dimensional. It follows from Theorems 1 and 4 that θ and θ' are both bijections. Let μ_α and μ be in $EP_G(Z)$ with $\theta(\mu_\alpha)$ converging to $\theta(\mu)$, but μ_α not converging to μ. We have

$$P_\alpha = T(\theta(\mu_\alpha)) \to P = T(\theta(\mu)),$$

hence if θ' is a homeomorphism,

$$\tau_\alpha = \theta'^{-1}(P_\alpha) \to \tau = \theta'^{-1}(P). \quad (1)$$

As

$$\theta(\mu_\alpha) = \pi\theta'(\tau_\alpha) = \theta\pi'(\tau_\alpha)$$

and θ is one-to-one, $\mu_\alpha=\pi'(\tau_\alpha)$ and similarly, $\mu=\pi'(\tau)$. From (1), $\mu_\alpha\to\mu$, a contradiction.

(G, Z) is said to be *free* if the isotropy groups are trivial. G is *amenable* if the regular representation of G weakly contains all of the irreducible representations (see [**3**, §18.3] and [**7**, §2.3]). Generalizing a result of Guichardet for semidirect products [**8**, p. 58], THEOREM 8. *If* (G, Z) *is free and* G *is amenable, then* π *is a homeomorphism.*

Letting (G, Z) be the transformation group of Theorem 3, G is amenable, and (G, Z) is free, as otherwise there would be a finite, thus closed orbit. As π is one-to-one, pr $\mathfrak{A}$ has only one point, and as π' is onto, $ET(\mathfrak{A})$ is uncountable.

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