EXCESSIVE FUNCTIONS OF CONTINUOUS TIME MARKOV CHAINS

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We consider transient continuous time Markov chains $P(t)$, with $P_{ij}'(0) = q_i\pi_{ij}$ for $i \neq j$ and $-q_i$ for $i = j$. We assume $0 < q_i < \infty$ for all i. Then $1/q_i$ is the mean time the process remains in state i, and π is the transition matrix of the imbedded jump process. We let q be a diagonal matrix with diagonal entries q_i.

A nonnegative function h is $P(t)$-excessive (invariant) if $h \geqq P(t)h$, $(h = P(t)h)$ for all t. It is π-superregular (regular) if $h \geqq \pi h$ $(h = \pi h)$. Our main results characterize the excessive functions of the minimal process in terms of q and π. These results can also be used to characterize excessive functions of certain nonminimal processes.

Let $P^{\min}(t)$ be the minimal substochastic Markov transition matrix with the given q and π. This is transient if and only if π is transient. Let $G^{\min}$ be the Green's function of $P^{\min}$. A nonnegative function h is $P^{\min}$-excessive if and only if it is π-superregular. Thus it may be represented as $h = r + Nf$, where r is π-regular and Nf is a π-potential. Since $Nf = G^{\min}(qf)$, it is also a (classical) $P^{\min}$-potential. However, r is in general not $P^{\min}$-invariant.

Let B be the Martin boundary of π. Then $r = c\int_B K(\cdot, x)d\mu^r(x)$, where μ^r is harmonic measure for π^r, the "h-process" with $h = r$. Let v_i be the probability that $P^{\min}$ started at i stops in finite time, and v^h be the corresponding vector for an h-process. We show that if h is minimal, π-regular, then $v^h = 0$ or 1. This defines a partition: $B = B_s + B_f$ into slow $(v^h = 0)$ and fast $(v^h = 1)$ boundary points. This partition differs from the customary active-passive classification, in that the criterion is applied to h-processes. If we write $r = r_s + r_f$, integrating separately over B_s and B_f, then r_s is $P^{\min}$-invariant and r_f is a $P^{\min}$-potential $(P^{\min}(t)r_f \to 0)$. Thus r is a $P^{\min}$-potential if and only if $\mu^r(B_s) = 0$. This potential is of class D (Meyer) if and only if μ^h is absolutely continuous with respect to μ. Thus we obtain many simple examples of nonclass D potentials.

These results are extended to a class of "instantaneous return processes" associated with a given π and q. We obtain both criteria for a function to be excessive (in terms of q, π, and the return probabilities), and a representation of the excessive functions.

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