ON CONVOLUTION AND FOURIER SERIES

BY JACK BRYANT

Communicated by A. Zygmund, September 8, 1966

In [4, pp. 108–114], Salem found that each function in $L_1(0, 2\pi)$ (or $C[0, 2\pi]$) can be represented as the convolution of a function in L (or C) with an even function in L with convex Fourier coefficients. We announce here a slight generalization of this theorem, and some related results which follow from a study of our methods. Detailed proofs will appear elsewhere [2].

We require the following notation: If f is a function, $(\tau_h f)(x) = f(x+h)$. B will denote a Banach space with norm $\|\cdot\|$. If $f \in L$, $S[f]$ denotes the Fourier series of f, $\{S_n\}$ the partial sums of $S[f]$ and $\{\sigma_n\}$ the $(C, 1)$ means of $S[f]$. $\|\cdot\|_1$ denotes the L_1-norm. If $\{\lambda_n\}$ is a sequence, $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$. We define Q to be the class of functions g with $S[g] = \lambda_0/2 + \sum \lambda_n \cos nx$, where $\Delta^2\lambda_n \geqq 0$ and $\lambda_n \to 0$. Note each function in Q is even, positive, integrable and differentiable on $(0, \pi)$. A will denote an absolute constant, not necessarily the same each time it appears.

THEOREM 1. *Suppose $S = \sum A_n$ is summable $(C, 1)$ to f in a Banach space B. Let ϕ be a positive increasing function with $\int_0^\infty 1/\phi(t)dt < \infty$. Let $\{\sigma_n\}$ be the $(C, 1)$ means of S; if $\{\lambda_n\}$ is a sequence such that $0 < \lambda_n \leqq \phi^{-1}(\|\sigma_n - f\|^{-1})$, $\Delta^2\lambda_n \leqq 0$ and $\lambda_n \uparrow \infty$, then the series $T = \sum \lambda_n A_n$ is summable $(C, 1)$ in B.*

THEOREM 2. *Let $B \subset L$ be a Banach space with $\|u\|_1 \leqq A\|u\|$ for each u in B, and suppose the $(C, 1)$ means of $S[f]$ are in B and $\|\sigma_n - f\| \to 0$. Then there exists $g \in Q$ and $h \in B$ such that $f = g * h$.*

THEOREM 3. *Let $f \in L$. Then $f = g * h$, where $g \in Q$ and $S[h]$ and $S[f]$ have, except for a set of measure zero, the same points of convergence.*

THEOREM 4. *Suppose $f \in L$, and let $\{\sigma_n\}$ be the $(C, 1)$ means of $S[f]$. If $\sum \|\sigma_k - f\|_1/k < \infty$ and if $\|\sigma_k - f\|_1 = o(1/\log k)$, then $S[f]$ converges almost everywhere.*

If we suppose more about B, Theorem 2 can be completed as follows:

THEOREM 5. *Let $B \subset L$ satisfy the following conditions: B is a Banach space and*

(1) *for each u in B, $\|u\|_1 \leqq A\|u\|$,*

(2) *for each u in B and each h, $\|\tau_h u\| \leqq A\|u\|$,*

(3) *for each f in* $L([0, 2\pi) \times [0, 2\pi))$,

$$\left\| \int_0^{2\pi} f(\cdot, t)dt \right\| \leqq A \int_0^{2\pi} \|f(\cdot, t)\| dt,$$

(4) *the partial sums of* $S[u]$ *are in* B *when* u *is in* B.

Then the following four conditions on an element f of B are equivalent:

(i) $\omega_1(f; t) = \sup\{\|\tau_h f - f\| : 0 \leqq |h| \leqq t\} = o(1)$,

(ii) $\omega_2(f; t) = \sup\{\|\tau_h f + \tau_{-h} f - 2f\| : 0 \leqq |h| \leqq t\} = o(1)$,

(iii) $S[f]$ *is summable* $(C, 1)$ *to* f *in* B,

(iv) $f = g * h$, $\quad h \in B$ *and* $g \in Q$.

Indication of Proofs. Theorem 1 is proved by summing the expression for the $(C, 1)$ means τ_n of $T = \sum \lambda_n A_n$ by parts twice and showing the sequence $\{\tau_n\}$ is Cauchy in B. We use Theorem 1 to obtain Theorem 2; the hypothesis $\|u\|_1 \leqq A\|u\|$ is necessary to show the series constructed is $S[h]$ for $h \in B$. To prove Theorem 3, we construct a seminorm on L so that convergence with respect to this seminorm implies a.e. convergence in the set of points of convergence of $S[f]$. We then use machinery developed for Theorem 1 plus the fact $\|\sigma_n - f\|_1 \to 0$ to obtain Theorem 3. Theorem 4 is an observation based on the fact that the numbers $\{1/\log n\}$ are convergence factors for Fourier series. (We note that in Theorem 4, σ_n may be replaced by s_n each time it appears.) Theorem 5 requires hypothesis (3), (4) for (ii)→(iii), (1) for (iii)→(iv) and (2) for (iv)→(i); (iii)→(iv) is Theorem 2, and the other proofs are elementary.

References

1. Jack Bryant, *Theorems relating convolution and Fourier series*, Ph.D. Thesis, Rice University, Houston, Texas, 1965.

2. ———, *On convolution and Fourier series*, Duke Math J. (to appear).

3. ———, *On convolution and moduli of continuity*, (to appear).

4. R. Salem, *Sur les transformations des series de Fourier*, Fund. Math. **33** (1939) 108–114.

Texas A&M University