DEDUCTION-PRESERVING "RECURSIVE ISOMORPHISMS" BETWEEN THEORIES[1,2]

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Introduction. In this work we discuss recursive mappings between theories which preserve deducibility, negation and implication. Roughly, we prove that any two axiomatizable theories containing a small fragment of arithmetic—this can be stated precisely—are "isomorphic" by a *primitive* recursive function mapping sentences onto sentences which also preserves deducibility, negation and implication (and hence theoremhood, refutability and undecidability). Also we prove between any two effectively inseparable theories formulated as applied predicate calculi there exists a "recursive isomorphism" preserving deducibility, negation and implication. In general, we cannot replace "recursive" by "primitive recursive" in the last result. From this we obtain a classification of all effectively inseparable theories into $\aleph_0$ equivalence classes. The unique maximal element is the equivalence class of those theories containing the small fragment of arithmetic referred to above. A more precise and detailed summary of the results—which answers some questions left open by Pour-El [4]—is given below following some notational remarks.

We believe that interest in the preservation of sentential connectives—especially implication—can be justified as follows. The preservation of implication implies the preservation of modus ponens and modus ponens is closely related to the deductive structure of the theories.

All theories considered in this paper will contain the propositional calculus. For definiteness we assume that implication and negation are the sole primitive propositional connectives: $A \lor B$ is an abbreviation for $\neg A \to B$; $A \cdot B$ is an abbreviation for $\neg(\neg A \lor \neg B)$. Furthermore in every section except section B the theories discussed will be formulated as applied predicate calculi. All theories considered will be both consistent and axiomatizable.

Notation. Let $\mathfrak{J}$ be a theory. Associated with $\mathfrak{J}$ is a *recursive* set W, the set of (Gödel numbers of) sentences and two recursively enumerable subsets of W, T the set of theorems and R the set of refutable sentences. We assume that W has an infinite complement $\overline{W}$.

In general, we identify a formula with its Gödel number. If a distinction is necessary it will be clear from the context.

DEFINITION 1. An applied predicate calculus $\mathfrak{J}$ is an effectively inseparable theory if (T, R) is an effectively inseparable (e.i.) pair of sets.

If $\mathfrak{J}$ is a propositional calculus then the concept of "sentence" may not have meaning since, for example $\mathfrak{J}$ may not possess variables. In this case we identify "sentence" with "formula." Thus Definition 1 applies in this case also.[3]

Survey of results. The results of A, B, and C below are consequences of some basic lemmas which are too complicated to state here.

   A. *Recursive mappings between applied predicate calculi.*

   I. If $\mathfrak{J}_1$ is consistent and $\mathfrak{J}_2$ is effectively inseparable *then* there is a 1-1 recursive function f^* mapping W_1 *into* W_2, $\overline{W}_1$ *onto* $\overline{W}_2$ such that for all B, C, in W_1

     (a) $f^*(B{\rightarrow}C) = f^*(B){\rightarrow}f^*(C)$

     (b) $f^*(\neg B) = \neg f^*(B)$

     (c) $B \vdash_{\mathfrak{J}_1} C$ *if and only if* $f^*(B) \vdash_{\mathfrak{J}_2} f^*(C)$.

Thus theorems are mapped into theorems, refutables into refutables and undecidables into undecidables.

   II. If $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are effectively inseparable *then* there is a 1-1 recursive function f^* mapping W_1 *onto* W_2, $\overline{W}_1$ *onto* $\overline{W}_2$ such that for all B, C in W_1

     (a) $f^*(B{\rightarrow}C) = f^*(B){\rightarrow}f^*(C)$.

     (b) $f^*(\neg B) = \neg f^*(B)$.

     (c) $B \vdash_{\mathfrak{J}_1} C$ *if and only if* $f^*(B) \vdash_{\mathfrak{J}_2} f^*(C)$.

   B. *Recursive mappings between applied propositional calculi.*

We show by example that II *does not hold for all applied propositional calculi.* Nevertheless given two effectively inseparable (e.i.) theories $\mathfrak{J}_1$ and $\mathfrak{J}_2$, it is possible to find a 1-1 recursive function g mapping W_1 *onto* W_2 preserving negation, deducibility and which *up to deductive equivalence preserves implication.* More precisely

   III. Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be two e.i. theories. There is a 1-1 negation-

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[3] Note that this extension of Definition 1 accords well with the original definition. For when $\mathfrak{J}$ is an applied predicate calculus which is e.i. by the definition then the *set of formulas* which are theorems is effectively inseparable from the set of refutable formulas.

preserving recursive function g mapping W_1 onto W_2 such that for all formulas B_1 and C_1 in $\mathfrak{I}_1$.

(a) $B_1 \vdash_{\mathfrak{I}_1} C_1$ if and only if $g(B_1) \vdash_{\mathfrak{I}_2} g(C_1)$.

(b) $\vdash_{\mathfrak{I}_2} g(B_1 {\rightarrow} C_1) \equiv . g(B_1) {\rightarrow} g(C_1)$

and for all formulas B_2 and C_2 in $\mathfrak{I}_2$.

(c) $\vdash_{\mathfrak{I}_1} g^{-1}(B_2 {\rightarrow} C_2) \equiv . g^{-1}(B_2) {\rightarrow} g^{-1}(C_2)$.

Note that as a consequence of III, theorems are mapped onto theorems, refutables are mapped onto refutables and undecidables are mapped onto undecidables.

In contrast I holds for the propositional calculus.

C. *Primitive recursive mappings between theories.*

For many mathematically interesting formal theories it is possible to strengthen results I and II by showing that f^* can be chosen to be primitive recursive. Suppose that $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are theories in standard formalization possessing a notation for the natural numbers and a binary predicate $\leq$. Suppose further that $\mathfrak{I}_i$ contains a subtheory $\mathfrak{I}_i'$ such that the following hold

(1) for all $n \vdash_{\mathfrak{I}_i'} x \leq \bar{n} \lor \bar{n} \leq x$,

(2) for all $n \vdash_{\mathfrak{I}_i'} x \leq \bar{n} {\rightarrow} . x = \bar{0} \lor \cdots \lor x = \bar{n}$,

(3) every primitive recursive function of one argument is definable in $\mathfrak{I}_i'$.

Then II holds for $\mathfrak{I}_1$ and $\mathfrak{I}_2$ with a primitive recursive f^*.

(An analogous statement may be made for I when $\mathfrak{I}_2$ contains a subtheory $\mathfrak{I}_2'$ satisfying (1), (2) and (3).)

Thus for example if $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are any two consistent axiomatizable extensions of the theory R of [7] , II holds for a primitive recursive f^*.

D. *A hierarchy of effectively inseparable theories.*

In contrast to the results of the preceding paragraph it is, in general, not possible to choose the f^* of I or II to be primitive recursive. For we prove

IV.[4] Let $\mathfrak{F}$ be a recursively enumerable class of general recursive functions. Then there exists an effectively inseparable theory $\mathfrak{I}_1$ in standard formalization such that no recursive function which witnesses the effective inseparability of $\mathfrak{I}_1$ is in $\mathfrak{F}$.

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[4] IV is a not too immediate consequence of the following result. Given an r.e. class $\mathfrak{F}$ of general recursive functions of a single argument we can find an effectively inseparable pair of sets (γ, δ) such that no recursive function which witnesses the effective inseparability of (γ, δ) is in $\mathfrak{F}$. A special case of this result, obtained by letting $\mathfrak{F}$ be the set of all primitive recursive functions, was obtained by McLaughlin. This result also generalizes results of Rogers [5] and Fischer (*Theory of Provable Recursive Functions*, MIT Doctoral Dissertation, 1962).

As an immediate corollary we obtain (where R is the theory of [7])

V. Given an r.e. class $\mathfrak{F}$ of general recursive functions, there exists a theory $\mathfrak{J}$ in standard formalization such that no 1-1 recursive function mapping W onto W_R, T onto T_R, R onto R_R preserving deducibility, negation and implication is in $\mathfrak{F}$.

IV gives rise to a classification of e.i. theories in standard formalization. Let $\mathfrak{F}_p$ be the set of all primitive recursive functions of one argument.

DEFINITION 2. $\mathfrak{J}_1$ is $\mathfrak{F}_p$-reducible to $\mathfrak{J}_2$ ($\mathfrak{J}_1 \leq_{\mathfrak{F}_p} \mathfrak{J}_2$) if there is an $f \in \mathfrak{F}_p$ mapping W_1 into W_2, T_1 into T_2, R_1 into R_2 preserving deducibility, negation and implication.

The reducibility relation of Definition 2 gives rise in a natural manner to an equivalence relation: $\mathfrak{J}_1 \equiv_{\mathfrak{F}_p} \mathfrak{J}_2$ if and only if $\mathfrak{J}_1 \leq_{\mathfrak{F}_p} \mathfrak{J}_2$ and $\mathfrak{J}_2 \leq_{\mathfrak{F}_p} \mathfrak{J}_1$. As mentioned earlier this equivalence relation partitions the e.i. theories into $\aleph_0$ classes with a unique maximum element. (Of course, the e.i. theories in standard formalization are also partitioned into $\aleph_0$ classes with a unique maximum element.)[5]

Results A, B, and C are obtained by generalizing the method of Myhill [3] (cf. [2] and [6]). Results in D are obtained by rate of growth arguments. A detailed account of the proofs is planned for a later publication.

We wish to thank Steven Orey for pointing out to us that we cannot replace Feferman's $\leqq$ by the $\leqq$ of Definition 2 in Theorem 6.4 of Feferman [1].

BIBLIOGRAPHY

1. S. Feferman, *Arithmetization of metamathematics in a general setting*, Fund. Math. 49 (1960), 35–92.

2. G. Kreisel, *Relative consistency and translatability*, J. Symbolic Logic 23 (1958), 108–109.

3. J. Myhill, *Creative sets*, Z. Math. Logik Grundlagen Math. 1 (1955), 97–108.

4. M. B. Pour-El, *"Recursive isomorphism" and effectively extensible theories*, Bull. Amer. Math. Soc. 71 (1965), 551–555.

5. H. Rogers, Jr., *Provable recursive functions*, Bull. Amer. Math. Soc. 63 (1957), 140.

6. R. Smullyan, *Theory of formal systems*, Annals of Mathematics Studies No. 47, Princeton Univ. Press, Princeton, N. J., 1961.

7. A. Tarski, A. Mostowski, R. Robinson, *Undecidable theories*, North Holland, Amsterdam, 1953.

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[5] Of course reducibility relations (and associated equivalence relations) can be obtained by replacing $\mathfrak{F}_p$ by many other r.e. classes. We will not digress to discuss this here.