ON RINGS OF OPERATORS[1]

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Let $\mathcal{H}$ be a complex Hilbert Space, $B(\mathcal{H})$ the ring of bounded operators on $\mathcal{H}$, E an abelian symmetric subring of $B(\mathcal{H})$ containing the identity which is closed in the weak operator topology, E_1 the commutant of E, and suppose E_1 has a cyclic vector ξ_0 which we normalize so that $|\xi_0|=1$. Diximier [1] has shown that E (respect. E_1), as a Banach space, is the dual of the Banach space R (respect. R_1) of all linear forms on E (respect. E_1) that are continuous in the ultra-strong topology of E (respect. E_1). In this note we show that every $T\in R$ is also continuous in the weak operator topology of E, from which it follows that a linear functional T on E is continuous in either the weak, ultraweak, strong, or ultrastrong topologies if and only if it is continuous in all four simultaneously. In the process, we obtain an integral representation for such T, which we later use in a theorem on centrally reducible positive functionals on E_1.

We denote the maximal ideal space of E by M, and for $A, B, \cdots \in E$, we denote the corresponding Gel'fand transforms by $a, b, \cdots$. Then $A\to a$ is an isometric isomorphism from E onto $C(M)$. Consequently, every bounded linear functional on E has the form

$$T(A) = \int_M a(m)\,d\nu(m), \tag{1}$$

where ν is a complex Borel measure on M uniquely determined by T. Of special interest are functionals of the form $A\to(A\xi, \xi)$, where ξ is a vector of $\mathcal{H}$. We denote by ν_ξ the measure corresponding to the vector ξ, and by μ the measure ν_{ξ_0}. Then the ν_ξ are all nonnegative, $\|\nu_\xi\| = |\xi|^2$, and it can be seen that $\nu_\xi \ll \mu$ for every ξ. The space M is extremely disconnected, and the measure μ has the special property that $C(M)\cong L_\infty(M, \mu)$ under the natural embedding.

THEOREM 1. *A linear functional T on E is ultrastrongly continuous if, and only if it is weakly continuous, and if and only if there exists a $\phi\in L_1(M, \mu)$ such that*

$$T(A) = \int_M a(m)\phi(m)\,d\mu(m) \tag{2}$$

for every $A\in E$.

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INDICATION OF PROOF. If T is ultrastrongly continuous, then T is norm continuous, hence we have the representation (1). Using the definitions of the ultrastrong topology and the above mentioned fact that $\nu_\xi \ll \mu$ for all ξ, it follows that $\nu \ll \mu$, hence we have the representation (2).

To prove that a T of the form (2) is weakly continuous, we may consider only the case where $\phi \geqq 0$. For $n=1, 2, \cdots$, we define $b_n(m) = \phi(m)$ if $n-1 \leqq \phi(m) < n$, $b_n(m) = 0$ otherwise. Then $b_n \in L_\infty(M, \mu)$ for each n. Using the fact that $C(M) \cong L_\infty(M, \mu)$, we may redefine each b_n on a null set $[\mu]$, and obtain the properties $b_n \in C(M)$, $b_n \cdot b_m = 0$ if $n \neq m$, $b_n \geqq 0$, $\sum b_n = \phi$ a.e. $[\mu]$. For each n, let $B_n \in E$ with Gel'fand transform b_n. Then $B_n \geqq 0$, and it follows readily that $AB_n^{1/2}\xi_0 \perp B_m^{1/2}\xi_0$ if $n \neq m$, $A \in E$, and furthermore

$$\begin{aligned} \sum \left| B_n^{1/2} \xi_0 \right|^2 &= \sum (B_n \xi_0, \xi_0) \\ &= \sum \int_M b_n(m)\, d\mu(m) \\ &= \int \phi(m)\, d\mu(m) < \infty. \end{aligned}$$

Setting $\eta = \sum_n B_n^{1/2}\xi_0$, we have for $A \in E$,

$$\begin{aligned} T(A) &= \int a(m)\phi(m)\, d\mu(m) \\ &= \sum_n \int a(m) b_n(m)\, d\mu(m) \\ &= \sum_n (AB_n\xi_0, \xi_0) \\ &= (A\eta, \eta), \end{aligned}$$

hence T is weakly continuous. Since weak continuity trivially implies ultrastrong continuity, the theorem follows.

Recall that a positive functional T_1 on E_1 is called centrally reducible if its restriction T to E has the following property: if T' is a positive functional on E such that $T' \leqq T$, then there exists a $B \in E$ such that $T'(A) = T(AB)$ for every $A \in E$.

THEOREM 2. *If T_1 is a weakly continuous positive functional on E_1, then T_1 is centrally reducible.*

PROOF. The restriction T of T_1 to E is also weakly continuous, hence it has the form (2), where $\phi \geqq 0$. If $0 \leqq T' \leqq T$, let ν' be the measure corresponding to T' in the representation (1). It follows that $0 \leqq \nu(E) \leqq \int_M X_E(m)\phi(m)d\mu(m)$, hence there exists $b \in L_\infty(M, \mu)$ such that $d\nu' = b\phi d\mu$. We may assume b continuous, and letting B be the operator of E having Gel'fand transform b, $T'(A) = T(AB)$ for arbitrary $A \in E$.

Reference

1. J. Dixmier, *Les algèbres d'opérateurs dans l'éspace Hilbertien* (*Algèbres de von Neumann*), Gauthier-Villars, Paris, 1957.

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