A NOTE ON THE MOMENTS OF THE NUMBER OF AXIS-CROSSINGS BY A STOCHASTIC PROCESS

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1. **Introduction.** A general formula, for moments of arbitrary order of the number of upcrossings of a level u by a stationary normal process in unit time, was obtained by Cramér and Leadbetter [1], using a combination of techniques due to Kac [3], and Ylvisaker [6]. Ylvisaker [7] has weakened the conditions of this result slightly by a proof which depends on interesting applications of martingale convergence theory and which may be applied also to nonstationary normal situations. In this note we give a somewhat different direct procedure, under the weakened conditions, for the calculation of these moments. This procedure gives an alternative to that of Ylvisaker [7] for normal processes, without the use of martingale theory, and may be also applied to nonnormal situations in the same way as the discussion in [4] for the first moment.

We shall here give the "counting procedure" used to obtain the number of upcrossings, sketching the derivation, and indicating the extension to nonnormal cases. A detailed proof along these lines (for the stationary normal case) will be given elsewhere (Cramér and Leadbetter [2]).

2. **A general result.** We shall consider a process $x(t)$ possessing, a.s., continuous sample functions and, for a given integer k, absolutely continuous $2k$-dimensional distributions with corresponding densities of the form $f_{t_1 \ldots t_{2k}}(x_1 \cdots x_{2k})$. There will be no loss of generality in considering the number N of upcrossings of the zero level by $x(t)$ in $0 \leqq t \leqq 1$, which is a well-defined random variable (cf. [4]).

For $\boldsymbol{t} = (t_1 \cdots t_k)$ lying in the k-dimensional unit cube, let m_r denote the unique integer such that $m_r/2^n \leqq t_r < (m_r+1)/2^n$. Write $E_n(\boldsymbol{t})$ for the k-dimensional cube whose sides are the intervals $[m_r/2^n, (m_r+1)/2^n)$. For $\epsilon > 0$, let $A_{n\epsilon}$ denote the set of all points $\boldsymbol{t}$ in the unit cube such that for all $\boldsymbol{s} = (s_1 \cdots s_k) \in E_n(\boldsymbol{t})$, we have $|s_i - s_j| > \epsilon$ whenever $i \neq j$, and write $\lambda_{n\epsilon}(\boldsymbol{t})$ for the characteristic function of the set $A_{n\epsilon}$. Finally let the random variable $\chi_{i,n} = 1$ if $x(i/2^n) < 0 < x[(i+1)/2^n]$, $\chi_{i,n} = 0$ otherwise. The following lemma

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gives the basic properties of the counting procedure used to obtain the factorial moments of N.

LEMMA. *Let* $M_{n\epsilon}=\sum'_{x_{i_1,n}\cdots x_{i_k,n}}\lambda_{n\epsilon}(i_1/2^n\cdots i_k/2^n)$, *where the summation is extended over all ordered sets of distinct integers* $i_1\cdots i_k$, $0\leq i_r\leq 2^n-1$. *Then, with probability one,*

(i) $M_{n\epsilon}$ *is nondecreasing as n increases or as* ϵ *decreases,*

(ii) $\lim_{n\to\infty}\lim_{\epsilon\to 0} M_{n\epsilon}=N(N-1)\cdots(N-k+1)$.

The proof of this lemma is accomplished by arguments extending those in [1, Part B]. From the monotonicity properties stated in (i) of the lemma it follows that the order of the ϵ and n-limits in (ii) may be interchanged. Hence, writing $M_k=\mathcal{E}N(N-1)\cdots(N-k+1)$, two applications of monotone convergence show that $M_k=\lim_{\epsilon\to 0}\lim_{n\to\infty}\mathcal{E}M_{n\epsilon}$. From the definition of $M_{n\epsilon}$ and a simple transformation of variables we thus have

$$(1)\quad M_k=\lim_{\epsilon\to 0}\lim_{n\to\infty}\sum{}'\lambda_{n\epsilon}(i_1/2^n\cdots i_k/2^n)\cdot P\{x_{i_r}<0<x_{i_r}+2^{-n}y_{i_r},\ r=1,2,\cdots,k\}$$

in which $x_r=x(r/2^n)$ and $y_r=2^n(x_{r+1}-x_r)$. In fact, the only nonzero terms in the sum on the right correspond to integers $i_1\cdots i_k$ satisfying $|i_r-i_s|>1$ for $r\neq s$. For such integer sets, the random variables $x_{i_1}\cdots x_{i_k}$, $y_{i_1}\cdots y_{i_k}$ possess a joint density. Write $\psi_{nt\epsilon}(x_1\cdots, x_k, y_1\cdots y_k)$ to be equal to this joint density for all $t=(t_1\cdots t_k)$ of $A_{n\epsilon}$ lying in the cube $E_n(i_1/2^n\cdots i_k/2^n)$, and $\psi_{nt\epsilon}=0$ outside such cubes. From (1) we then obtain the following result by straightforward calculation.

THEOREM. *For the process* $x(t)$ *considered, the kth factorial moment of N is*

$$(2)\quad M_k=\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_0^1\cdots\int dt\int_0^\infty\cdots\int dy\int_{-y_1}^0\cdots\int_{-y_k}^0\cdot\psi_{nt\epsilon}(2^{-n}x_1\cdots 2^{-n}x_k, y_1\cdots y_k)dx.$$

3. Normal processes and generalizations. It can easily be seen that the above assumptions are satisfied for a (separable) stationary normal process $x(t)$ whose covariance function has a finite second derivative at the origin. Further if $p_t(x, y)$ denotes the joint density for $x(t_1)\cdots x(t_k)$ and the quadratic mean derivatives $x'(t_1)\cdots x'(t_k)$, it can be shown (cf. [5]) by convergence of covariances that $\psi_{nt}(2^{-n}x_1\cdots 2^{-n}x_k, y_1\cdots y_k)\to p_t(0, y)$ as $n\to\infty$, for all t in the region $D(\epsilon)=\lim_{n\to\infty} A_{n\epsilon}$. It can be shown by dominated convergence that the n-limit in (2) can be taken inside all the integral signs, and it is then an easy application of monotone convergence as $\epsilon\to 0$ to obtain the result:

$$(3)\quad M_k = \int_0^1 \cdots \int dt \int_0^\infty \cdots \int y_1 \cdots v_k p_t(0, y)dy \leqq \infty.$$

Certain nonnormal processes may be treated in a similar way from (2) to obtain (3), (see, for example, the derivation of $\mathcal{E}N$ in [4] for the *envelope* of a stationary normal process.) In general we may obtain a result corresponding to that given in [4] for the mean. To that end write, for $t=(t_1 \cdots t_k)$,

$$g_{t,\tau}(x, y) = \tau^k f_{t_1\cdots t_k,\ t_1+\tau\cdots t_k+\tau}(x_1 \cdots x_\kappa, x_1 + \tau y_1 \cdots x_k + \tau y_k).$$

That is $g_{t,\tau}$ is the joint density for the $x(t_i)$ and the incrementary ratios $(x_{t_i+\tau}-x_{t_i})/\tau$. Then we have the following result.

THEOREM. *Consider points $t=(t_1 \cdots t_k)$ such $t_i\neq t_j$ for $i\neq j$ and suppose that,*

(i) $g_{t\tau}(x, y)$ *is continuous in* (t, x) *for each* y, τ,

(ii) *For each* $\epsilon>0$, $g_{t\tau}(x, y)\to p_t(x, y)$ *as* $\tau\to 0$ *uniformly in* (t, x) *for* $t\in D(\epsilon)$ *and each* y.

(iii) *For each* $\epsilon>0$, *there is a function* $h_\epsilon(y)$ *such that for* $t\in D(\epsilon)$,

$$g_{t,\tau}(x, y) \leqq h_\epsilon(y) \quad \text{and} \quad \int_0^\infty \cdots \int y_1 \cdots y_k h_\epsilon(y)dy < \infty.$$

Then (3) *holds for the process* $x(t)$.

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