COCOMMUTATIVE HOPF ALGEBRAS WITH ANTIPODE

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We shall describe the structure of a certain kind of Hopf algebra over an algebraically closed field k of characteristic p, namely those Hopf algebras whose coalgebra structure is commutative and which have an antipodal map $S: H \to H$. (See below for definitions.) Such a Hopf algebra turns out to be of the form $kG \# U$, the smash product of a group algebra with a Hopf algebra whose coalgebra structure is "like" that of a universal enveloping algebra. If $p=0$ the second factor actually is a universal enveloping algebra.

For $p>0$, we generalize the Birkhoff-Witt theorem by introducing the notion of divided powers. These also play a role in the theory of algebraic groups where certain sequences of divided powers correspond to one parameter subgroups. The divided powers appear in a "Galois Theory" for all finite normal field extensions.

The structure theory of Z_2-graded coanticommutative Hopf algebras is similar, and mentioned below.

Lemma 1, Theorem 1, its generalization to the graded case, and Theorem 2 are unpublished results of B. Kostant, whose guidance we gratefully acknowledge.

1. H is a cocommutative Hopf algebra with multiplication m, augmentation ϵ and diagonal d.

DEFINITION. An element $g \in H$ is *grouplike* if $dg = g \otimes g$ and $g \neq 0$.

LEMMA 1. *The set G of grouplike elements of H form a multiplicative semigroup whose elements are linearly independent in H. For each $g \in G$ there exists a unique maximal coalgebra $H^g \subset H$ whose only grouplike element is g. $H \cong \oplus H^g$ as a coalgebra, and $H^g H^h \subset H^{gh}$.*

DEFINITION. $S: H \to H$ is an *antipode* if

$$m \circ (I \otimes S) \circ d = \epsilon = m \circ (S \otimes I) \circ d.$$

THEOREM 1. *If H has an antipode G is a group and $S(g) = g^{-1}$. If e is the identity of G, $H^g = gH^e = H^e g$, and $H \cong kG \# H^e$ as a Hopf algebra.*

REMARK. Since $g^{-1}H^e g = H^e$, the elements of G act as Hopf algebra automorphisms of H^e and so we can form the smash product $kG \# H^e$.

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(As a coalgebra this is $kG\otimes H^e$, $(1\otimes h)(g\otimes 1)=(g\otimes g^{-1}hg)$ $g\in G$, $h\in H^e$.)

If F is a cocommutative Hopf algebra with one grouplike element, G a group of Hopf algebra automorphisms of F then $kG \# F$ has a unique antipode.

In the Z_2-graded coanticommutative situation, $G \subset H_0$, $H^e =(H^e\cap H_0)\oplus(H^e\cap H_1)$. If H has an antipode, G is a group and $H\cong kG \# H^e$ as a graded Hopf algebra.

2. We now determine the structure of H^e, i.e. we consider a Hopf algebra H with one grouplike element.

THEOREM 2. *If $p=0$, H is the universal enveloping algebra of the Lie algebra L (under $[\ ,\ ]$), where*

$$L = \{x\in H \mid dx = x\otimes 1 + 1\otimes x\}.$$

DEFINITION. For arbitrary p the elements of L are called *primitive*. If $p>0$, L is a restricted Lie algebra but H is not necessarily its restricted universal enveloping algebra. However, using the Birkhoff-Witt theorem we can get a form of Theorem 2 which does generalize to $p>0$. Namely it says for $p=0$, $H=\otimes C_\gamma$ as a coalgebra, where C_γ is the subspace of H spanned by the elements ${}^el_\gamma=l_\gamma^e/e!$ $e=0, 1, \cdots$ and $\{l_\gamma\}$ is a basis for L. Note that C_γ is a coalgebra because $d\,{}^el_\gamma =\sum_0^e {}^il_\gamma\otimes {}^{e-i}l_\gamma$.

DEFINITION. A finite or infinite sequence of elements $1={}^0l$, ${}^1l$, ${}^2l, \cdots$ is called a *sequence of divided powers of* ${}^1l$ if $d\,{}^nl=\sum_0^n {}^il\otimes{}^{n-i}l$.

Given an indeterminate x, let H_x^∞ be the Hopf algebra with a basis of indeterminates ${}^ix$, $i=0, 1, 2, \cdots$, the algebra structure is determined by ${}^ix\,{}^jx=\binom{i+j}{j}x^{i+j}$ and the coalgebra structure is determined by ${}^0x$, ${}^1x, \cdots$, which is a sequence of divided powers of ${}^1x$. If $p>0$ we let H_x^n be the sub-Hopf algebra spanned by ${}^0x$, ${}^1x, \cdots, {}^{p^n-1}x$.

Let $H'=\text{Hom}(H, k)$ have the algebra structure "transpose" to the coalgebra structure of H. Thus for a', $b'\in H'$, $a' * b'$ is the map $(a'\otimes b')\circ d\colon H\to k$. H' is a commutative algebra since H is cocommutative.

THEOREM 3. *For $p>0$, let $I^n\subset H'$ be the ideal generated by $\{a'\in H' \mid a'^{p^n}=0\}$. If the sequence of ideals $I^1\subset I^2\subset\cdots$ terminates, then $H\cong\otimes H_x^{n_x}$ as a coalgebra, for some set of elements $\{x\}$ and positive integers (or ∞), $\{n_x\}$.*

If $I^1=0$, $H\cong\otimes H_x^\infty$ as a coalgebra, where we may choose $\{x\}$ to be a basis for L.

If $I^1 = \{a' \in H' \mid a'(1) = 0\}$, *then* H *is the restricted universal enveloping algebra of* L. *So* $H \cong \otimes H'_x$ *as a coalgebra, where* $\{x\}$ *is a basis for* L.

The techniques involved in proving Theorem 3 yield information about sequences of divided powers lying above an element of L. For example, $l \in L$ is orthogonal to I^n if and only if l lies in a sequence of divided powers ${}^0l$, ${}^1l = l$, ${}^2l, \cdots {}^{p^{n+1}-1}l$.

In the coanticommutative situation the Hopf algebra H contains a unique maximal sub Hopf algebra $F \subset H_0$. Theorem 2 or 3 applies to F. If $L_0 = L \cap H_0$ and $L_1 = L \cap H_1$ then $L = L_0 \oplus L_1$ and L is a graded Lie algebra. If ΛL_1 is the exterior algebra on L_1 then $H \cong F \otimes \Lambda L_1$ as a coalgebra. If $p = 0$, H is the graded universal enveloping algebra of L.

BIBLIOGRAPHY

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