COHOMOLOGY OF ALGEBRAIC GROUPS AND INVARIANT SPLITTING OF ALGEBRAS[1,2]

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1. **Introduction.** Let A be an algebra, over a field F, assumed at first to be associative and finite-dimensional over F. Let R be the radical of A, C the center of A. Assume A/R separable, so that A possesses maximal separable subalgebras (Wedderburn factors) S for which $A=S+R$, $S\cap R=0$. Let G be a group of automorphisms and antiautomorphisms of A. We will discuss the existence and uniqueness of G-invariant Wedderburn factors in terms of various cohomology groups of G. In general, the cohomology is that of abstract groups. However, the conditions given will be compatible with taking the algebraic hull of G (in the Zariski topology with respect to F), so that we can assume G is an algebraic group and the cohomology is rational. We will outline here how the cohomology enters. Details will appear elsewhere. See [3], [4], [5] for a general background of the question.

2. **Existence.** We first assume $R^2=0$. Let S be any maximal separable subalgebra. If $g\in G$, then Sg is another maximal separable subalgebra, so by the Malcev theorem, $Sg=SC_{1-z(g)}$, where C_w is conjugation by w. $z(g)$ is in R, but is uniquely determined modulo $R\cap C$, so that we consider z as a function from G to the vector space $R/R\cap C$. We consider $R/R\cap C$ as a G-module in the obvious way, except that the antiautomorphisms in G act via their negatives. Then a technical calculation will show that $z\in Z^1(G, R/R\cap C)$, i.e., $z(gh)=z(g)\cdot h+z(h)$. Hence if $H^1(G, R/R\cap C)=0$, there is an x in R such that $z(g)=x-x\cdot g+R\cap C$. A technical calculation will then show that SC_{1-x} is a G-invariant maximal separable subalgebra.

Now we consider the general case $R^2\neq 0$. The action of G on all modules will be the obvious ones, except that the antiautomorphisms in G will act via their negatives. We consider A/R^2. The condition for the case $R^2=0$ above now becomes $H^1(G, R/\{x\in R \mid [A,x]\subseteq R^2\})=0$ where $[A, x]=\{[a, x]=ax-xa \mid a\in A\}$. If this holds, then $A=S_1+R$, S_1 a G-invariant subalgebra, $S_1\cap R\subseteq R^2$. S_1 has radical R^2, and we next consider S_1/R^4. The condition now is

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$H^1(G, R^2/\{x\in R^2 | [S_1, x]\subseteq R^4\})=0$. This yields $A=S_2+R$, S_2 a G-invariant subalgebra, $S_2\cap R\subseteq R^4$. Let $R^{2^n}\neq 0$, $R^{2^{n+1}}=0$. Then the conditions become $H^1(G, R^{2^i}/\{x\in R^{2^i} | [x, S_i]\subseteq R^{2^{i+1}}\})$ for $i=0, 1, \cdots, n$, where $S_0=A, S_1, \cdots, S_n$ are G-invariant subalgebras as indicated. The next step yields S_{n+1} as a G-invariant maximal separable subalgebra.

3. **Applications.** All the modules considered are rational modules for the algebraic hull of G, and the cocycles are rational functions. Hence we may assume G is an algebraic group. If G is reductive, then the rational cohomology $H^1(G, M)=0$ for M a rational G-module. This follows from an argument in [1] as follows: Let $W=F\oplus M$, $f\in Z^1(G, M)$. Let G act on W by $(a, m)g=(a, mg+f(g))$. W is completely reducible since G acts rationally on it. Let C be a G-complement to M in W. C has a unique element $(1, x)$, x in M. Applying $g\in G$ yields $f(g)=x-xg$. This argument shows that A possesses G-invariant maximal separable subalgebras if the algebraic hull of G is a reductive algebraic group. In particular, it holds if F has characteristic zero and G is completely reducible (see [2]).

The cohomology conditions are well-known if G is a finite group of order not divisible by the characteristic of F.

Note that $\{x\in R^{2^i} | [x, S_i]\subseteq R^{2^{i+1}}\}$ is a Lie ideal in S_i. This indicates that similar results hold for Lie algebras over fields of characteristic zero.

By inducting on the degree of nilpotency of R, rather than on the dimension of A, we note that the cohomology conditions (for abstract groups) will suffice for infinite-dimensional algebras (with nilpotent radicals), provided the algebras involved possess Wedderburn principal decompositions which satisfy the Malcev theorem.

See also [5] for additional results concerning existence.

4. **Uniqueness.** Here the aim is to prove that any two G-invariant maximal separable subalgebras S and T are G-orthogonally conjugate (see [4] for definitions). Again, G will act naturally on the modules which arise, except that the antiautomorphisms act via their negatives. Also let characteristic $F\neq 2$.

First let $R^2=0$, z in R so that $SC_{1-z}=T$. A technical calculation yields $z-z\cdot g$ in $R\cap C$, and $f(g)=z-z\cdot g$ satisfies f in $Z^1(G, R\cap C)$. If $H^1(G, R\cap C)=0$, then there is an x in $R\cap C$ so $z-z\cdot g=x-x\cdot g$. Then $z-x$ is G-skew (called G-symmetric in [4]) so $1-z+x$ is G-orthogonal, and $SC_{1-z+x}=T$.

Now consider the general case. The cohomology condition for A/R^2 becomes $H^1(G, \{x\in R | [T, x]\subseteq R^2\}/R^2)=0$. Then this yields

z_1 in R so that z_1+R^2 is G-skew in R/R^2 and $SC_{1-z_1}+R^2=T+R^2$. Define $f_1(g)=\frac{1}{2}(z_1\cdot g-z_1)$. Then $f_1\in Z^1(G, R^2)$. If $H^1(G, R^2)=0$, we get an x_1 in R^2 so $f_1(g)=x_1-x_1\cdot g$. Let $y_1=-x_1-z_1/2$. Then y_1 is G-skew and $y_1+R^2=-z_1/2+R^2$. Let $u_1=-2y_1(1-y_1)^{-1}$. Then $1-u_1$ is G-orthogonal and $SC_{1-u_1}+R^2=T+R^2$.

Repeating these arguments, two sets of conditions emerge. They are (1) $H^1(G, \{x\in R^{2^i} \mid [T, x]\subseteq R^{2^{i+1}}\}/R^{2^{i+1}})=0$, $i=0, 1, \cdots, n$, and (2) $H^1(G, R^{2^i})=0, i=1, \cdots, n$. The first set yields G-skew cosets. The second set enables one to lift out suitable G-skew elements out of which are built G-orthogonal elements using a Cayley transform. Finally $(1-u_1)(1-u_2)\cdots(1-u_n)(1-u_{n+1})$ conjugates S into T.

The same general remarks in §3 apply here also. However, we point out that for finite-dimensional algebras the conclusion can be proved for any completely reducible group, even if the characteristic is not zero (but not two), as in [4].

References

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