A RELATION BETWEEN MOMENT GENERATING FUNCTIONS AND CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS

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Communicated by M. Loève, July 20, 1966

Let X_N for $N=0, \pm 1, \cdots$ be independent random variables with finite first absolute moments; let $A_N=\{a_{N,k}: k=0, \pm 1, \cdots\}$; let $\|A_N\|_\infty = \sup_k |a_{N,k}|$ and $\|A_N\|_p = [\sum_k |a_{N,k}|^p]^{1/p}$ for $1 \leqq p < \infty$; let $S_N = \sum_k a_{N,k}(X_k - EX_k)$; and let p and q be numbers in $[1, \infty]$ satisfying $1/p+1/q=1$.

THEOREM. *Suppose there exist positive constants M, γ, and $1 \leqq p \leqq 2$ such that for $0 < x < \infty$ and all values of k*

$$P\{|X_k - EX_k| \geqq x\} \leqq \int_x^\infty M \exp(-\gamma t^p) dt. \tag{1}$$

Suppose $\|A_N\|_2$ and $\|A_N\|_q$ are finite for all N. Then

$$T_N = \lim_{\alpha \to -\infty;\, \beta \to \infty} \sum_{k=\alpha}^{\beta} a_{N,k}(X_k - EX_k)$$

exists as an almost sure limit for each N and there exist positive constants C_1 and C_2 such that for every $\epsilon > 0$

$$P\{T_N \geqq \epsilon\} \leqq \exp\left[-\min\left\{C_1\left(\frac{\epsilon}{\|A_N\|_2}\right)^2, C_2\left(\frac{\epsilon}{\|A_N\|_q}\right)^p\right\}\right].$$

The constants C_1 and C_2 which are obtained depend only on M, γ, and p. They do not depend in any other way on the distribution of the X_k's and they do not depend on the coefficient sequences A_N.

When $p=1$ the condition (1) is equivalent to the existence of constants $T>0$ and $C>0$ such that $E\exp(tX_k) \leqq \exp(Ct^2)$ for all k and all $|t| < T$; when $1 < p \leqq 2$ it is equivalent to the existence of a constant $C>0$ such that $E\exp(tX_k) \leqq \exp[C(t^2+|t|^q)]$ for all k and t.

$$\text{If } p = 1 \text{ and } a_{N,k} = 1/N \quad \text{for } k = 1, \cdots, N,$$
$$= 0 \quad \text{otherwise,}$$

[1] Research partially supported by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant Nr AF-AFOSR-746-65.

then this theorem reduces to the well known result (see [1] and [2]) giving exponential convergence rates in the law of large numbers; that is, the theorem guarantees $0 \leqq \rho_\epsilon < 1$ such that $P\{T_N \geqq \epsilon\} \leqq (\rho_\epsilon)^N$ for all N.

If $p=1$ and $\|A_N\|_1 \leqq M < \infty$ for all N, then since $\|A_N\|_2^2 \leqq \|A_N\|_1 \times \|A_N\|_\infty$ we see that there exists $0 \leqq \rho_\epsilon < 1$ such that $P\{T_N \geqq \epsilon\} \leqq [\rho_\epsilon]^{1/\|A_N\|_\infty}$. The assumption made here about the distributions of the X_k's is equivalent to that made in (1) of Theorem 1 of [3]. Thus we obtain Theorem 1 of [3] as a corollary to the theorem given above. We actually obtain a stronger result than Theorem 1 of [3] since it is not necessary for $\|A_N\|_1$ to even be finite for our theorem to hold.

If $p=2$, then $\|A_N\|_2 = \|A_N\|_q$ and we obtain $P\{T_N \geqq \epsilon\} \leqq [\rho_\epsilon]^{1/\|A_N\|_2^2}$ for some $0 \leqq \rho_\epsilon < 1$. This is essentially Chow's Lemma 2 in [4]. We can obtain generalized versions of his Theorems 1 and 2 from our theorem.

Note that no improvement can be obtained by taking $p > 2$, and in fact not even by assuming a uniform bound on the X_k's. The Central Limit Theorem seems to provide a bound on the rate of convergence obtained in this theorem.

A continuous case analogue similar to the theorem of [5] and proofs will be published elsewhere.

References

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