MULTIPLICATION IN GROTHENDIECK RINGS OF INTEGRAL GROUP RINGS

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1. **Introduction.** Let G be a finite group, Z the ring of rational integers, and form the Grothendieck ring $K^0(ZG)$ of the integral group ring ZG. Swan [4] has described multiplication in $K^0(ZG)$ when G is cyclic of prime power order. The purpose of this note is to present results which describe multiplication in $K^0(ZG)$ when G is cyclic or elementary abelian. Full details will appear elsewhere.

Let Q denote the rational field, and recall that the elements of $K^0(QG)$ are Z-linear combinations of symbols $[M^*]$, where M^* ranges over all finitely-generated left QG-modules, and similarly for $K^0(ZG)$. We define a ring epimorphism $\theta\colon K^0(ZG)\to K^0(QG)$ by $\theta[M] = [Q\otimes_Z M]$, and call any linear mapping $f\colon K^0(QG)\to K^0(ZG)$ such that $\theta f=1$ a *lifting map* for $K^0(ZG)$. Since the Jordan-Hölder Theorem holds for QG-modules, $K^0(QG)$ is the free abelian group with basis $\{[M_i^*]\colon 1\leqq i\leqq m\}$, where $\{M_i^*\colon 1\leqq i\leqq m\}$ is a full set of nonisomorphic irreducible QG-modules. Swan [4] has shown that to describe multiplication in $K^0(ZG)$ it suffices to describe the products $f[M_i^*]\cdot f[M_j^*]$, for $1\leqq i, j\leqq m$, and $f[M_i^*]x$, for $1\leqq i\leqq m$ and $x\in\ker\theta$.

2. **Statement of results.** Let G be cyclic of order n with generator g. For each s dividing n, ζ_s will denote a primitive sth root of unity, and Z_s will denote the ZG-module $Z[\zeta_s]$ on which g acts as ζ_s. Similarly, Q_s will denote the QG-module $Q(\zeta_s)$. Then $K^0(QG)$ is the free abelian group with basis $\{[Q_s]\colon s|n\}$, and $f\colon K^0(QG)\to K^0(ZG)$ by $f[Q_s]=[Z_s]$ is a lifting map. Swan [4] has shown that f is a ring homomorphism. Also, for each s dividing n, G_s will denote the quotient group of G of order s, and if $t|s$, $N_{s/t}$ will denote the norm from Q_s to Q_t. By the results of Heller and Reiner [2],

$$\ker\theta = \left\{\sum_{s|n} ([A_s] - [Z_s])\colon A_s = Z_s\text{-ideal in } Q_s\right\}.$$

THEOREM 1. *Multiplication in $K^0(ZG)$ is given by the formula*

$$[ZG_r]([A_s] - [Z_s]) = \sum_d ([N_{s/s'}(A_s)Z_d] - [Z_d]),$$

for all r, s dividing n, where $s'=s/(r,s)$ and d ranges over all divisors of $[r,s]$ such that $([r,s]/d, s')=1$.

THEOREM 2. *If G is an elementary abelian group, multiplication in $K^0(ZG)$ can be explicitly determined.*

We remark that it is possible to give formulas which describe multiplication in $K^0(ZG)$ when G is elementary abelian. These formulas will not be included here.

3. **Proof of Theorem 1.** We first suppose that $r=p^a$, for some prime p and nonnegative integer a, and write $s=p^b t$, $(p, t)=1$. If $a=0$ or $b=0$, the theorem is trivial. Let $\hat{Z}=Z_s/A_s$ and for each t dividing s, let $\hat{Z}\langle\bar{\zeta}_t\rangle$ denote the ZG-module $\hat{Z}$ on which g acts as ζ_t reduced modulo A_s. It suffices to find $M=ZG_{p^a}\otimes_Z \hat{Z}$. Since $ZG_{p^a}\cong Z[x]/(x^{p^a}-1)$, $M\cong\hat{Z}[x]/(x^{p^a}-1)$. If $a\leqq b$, then in $\hat{Z}[x]$, $x^{p^a}-1=\prod_k (x-\bar{\zeta}_{p^a}^k)$, $1\leqq k\leqq p^a$, and thus $M\cong\sum_k\hat{Z}\langle\bar{\zeta}_p^{b_t}\bar{\zeta}_{p^a}^k\rangle$. A calculation with norms now yields the desired result. If $a>b$, then $x^{pa}-1$ factors in $\hat{Z}[x]$ as follows: $x^{pa}-1=\prod_k (x-\bar{\zeta}_{p^b}^k)\prod_{i,j} (x^{p^{i-b}}-\bar{\zeta}_{p^b}^j)$, where $1\leqq k\leqq p^b$, $b+1\leqq i\leqq a$, and $1\leqq j\leqq p^b$ with $(p, j)=1$. Therefore

$$M\cong\sum_k \hat{Z}\langle\bar{\zeta}_p^{b_t}\bar{\zeta}_{p^b}^k\rangle+\sum_{i,j} (Z_{p^i t}/A_s Z_{p^i t})\langle\bar{\zeta}_{p^b t}\bar{\zeta}_{p^i}^j\rangle,$$

where $(Z_{p^i t}/A_s Z_{p^i t})\langle\bar{\zeta}_p^{b_t}\bar{\zeta}_{p^i}^{j_i}\rangle$ denotes the ZG-module $Z_{p^i t}/A_s Z_{p^i t}$ on which g acts as $\bar{\zeta}_p^{b_t}\bar{\zeta}_{p^i}^{j_i}$. Again, a calculation with norms will yield the desired result. This proves the theorem for the case $r=p^a$. The general case follows by the use of induction on the number of distinct prime divisors of r.

4. **Proof of Theorem 2.** In order to prove Theorem 2, we need several lemmas.

LEMMA 1. *Let G be an abelian group, F an algebraic number field which is a splitting field for G, and R the ring of algebraic integers of F. Then multiplication in $K^0(RG)$ can be explicitly determined.*

Let G be an elementary abelian group and write $G=G_1\times\cdots\times G_k$, where G_i is cyclic of order p with generator g_i, for $1\leqq i\leqq k$. Let ζ be a primitive pth root of unity, $F=Q(\zeta)$, $R=Z[\zeta]$, and denote by $F\langle a_1,\cdots, a_k\rangle$ the FG-module F on which g_i acts as ζ^{a_i}, where $1\leqq a_i\leqq p$ for $1\leqq i\leqq k$. Similarly, if A is any R-ideal in F, $A\langle a_1,\cdots, a_k\rangle$ will denote the RG-module A on which g_i acts as ζ^{a_i}. Note that, by restriction of operators, $F\langle a_1,\cdots, a_k\rangle$ and $A\langle a_1,\cdots, a_k\rangle$ are QG- and ZG-modules, respectively. It is easy to prove that the QG-modules of form $F\langle p,\cdots, p, 1, a_{j+1},\cdots, a_k\rangle$, where $1\leqq j\leqq k$, together with the trivial module Q, form a full set of nonisomorphic irreducible QG-modules.

Define $\psi: K^0(ZG)\to K^0(RG)$ by $\psi[Y]=[R\otimes_Z Y]$, where $R\otimes_Z Y$ is an RG-module with action of R given by $r'(r\otimes y)=r'r\otimes y$, for all

$r' \in R$, and action of G given by $g(r \otimes y) = r \otimes gy$, for all $g \in G$. Similarly, define $\eta: K^0(QG) \to K^0(FG)$ by $\eta[Y^*] = [F \otimes_Q Y^*]$.

LEMMA 2. *ψ and η are ring homomorphisms and the following diagram commutes and is exact:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \theta_R & \longrightarrow & K^0(RG) & \xrightarrow{\theta_R} & K^0(FG) & \longrightarrow & 0 \\ & & \uparrow \psi & & \uparrow \psi & & \uparrow \eta & & \\ 0 & \longrightarrow & \ker \theta_Z & \longrightarrow & K^0(ZG) & \xrightarrow{\theta_Z} & K^0(QG) & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & 0 & & \end{array}$$

Let $\Phi_p(x)$ denote the cyclotomic polynomial of order p. If we apply ψ to $[A\langle p, \cdots, p, 1, a_{j+1}, \cdots, a_k\rangle] \in K^0(ZG)$, we note that $\Phi_p(g_j)$ annihilates $R \otimes_Z A\langle p, \cdots, p, 1, a_{j+1}, \cdots, a_k\rangle$. Since $\Phi_p(x)$ splits into linear factors in $R[x]$, this gives us a composition series for $R \otimes_Z A\langle p, \cdots, p, 1, a_{j+1}, \cdots, a_k\rangle$. If we denote by $A^{(t)}$ the ideal conjugate to A under the Q-automorphism of F which takes ζ into ζ^t, we thus obtain

LEMMA 3. $\psi[A\langle p, \cdots, p, 1, a_{j+1}, \cdots, a_k\rangle] = \sum_t [A^{(t)}\langle p, \cdots, p, t, ta_{j+1}, \cdots, ta_k\rangle]$, *where* $1 \leq t \leq p-1$.

We now use the formulas for $\ker \theta_Z$ and $\ker \theta_R$ obtained by Heller and Reiner [2], and our formula for $\psi[A\langle p, \cdots, p, 1, a_{j+1}, \cdots, a_k\rangle]$, to show that $\psi: \ker \theta_Z \to \ker \theta_R$ is monic. Lemma 2 then implies that $\psi: K^0(ZG) \to K^0(RG)$ is monic. Now define $f_R: K^0(FG) \to K^0(RG)$ by $f_R[F\langle a_1, \cdots, a_k\rangle] = [R\langle a_1, \cdots, a_k\rangle]$. It is clear that f_R is a lifting map for $K^0(RG)$, and it is easy to show that f_R is a ring homomorphism. Since ψ is monic, we may define $f_Z = \psi^{-1} f_R \eta$. Then f_Z is a lifting map for $K^0(ZG)$ and is a ring homomorphism. Finally, since F is a splitting field for G, multiplication in $K^0(RG)$ is known by Lemma 1, and hence multiplication in $K^0(ZG)$ can be explicitly determined by the use of the monomorphism ψ. This completes the proof of Theorem 2.

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