SUMS OF ULTRAFILTERS

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The main result, an estimate of the cardinality of the set of all ultrafilters producing a given type of ultrafilter (see definition 1.4 and Theorem C in 1.4), is illustrated by a proof of nonhomogeneity of $\beta N - N$ (see 2.1) without using the continuum hypothesis, and by an exhibition of the following two examples.

THEOREM A. *For each positive integer n there exists a space X such that X^n is countably compact but X^{n+1} is not.*

THEOREM B. *There exists a space Y such that each finite product Y^n is countably compact but $Y^{\aleph_0}$ is not.*

By a space we mean a separated uniformizable topological space; and Z^m stands for the product of any constant family $\{Z \mid a \in A\}$ such that the cardinal of A is m.

In our examples the spaces X^{n+1} in A and $Y^{\aleph_0}$ in B are not pseudocompact. An exhibition of A and B with countably compact replaced by pseudocompact is done in [3]; it does not require Theorem C. Trivial examples of spaces with properties in A and B do not seem to be available.

Observe the proof of A and B reduces to the following.

THEOREM A′. *For each positive integer n there exist spaces $X(1), \cdots, X(n+1)$ such that the product of any family $\{X(k_i) \mid i = 1, \cdots, n\}$ is countably compact but the product $\{X(j) \mid j \leqq n+1\}$ is not countably compact.*

THEOREM B′. *There exists a sequence $\{Y(j)\}$ of spaces such that the product of any finite subfamily is countably compact but the product of $\{Y(j)\}$ is not.*

Indeed, for an X in A take the sum of spaces $X(j)$ with properties in A′. For Y in B take a one-point countable-compactification of the sum of a sequence $\{Y(j)\}$ with properties in B′; then the product of $\{(j) \times Y(j)\}$ is a closed subspace of Y.

REMARK. In addition, we shall exhibit $\{Y(j)\}$ such that the product of any proper subfamily (e.g. $\{Y(j) \mid j \geqq 2\}$) is countably compact. On the other hand there exists a sequence $\{Y(j)\}$ such that the product of a subfamily is countably compact if and only if the subfamily is finite.

In what follows N denotes the set and the discrete space of natural numbers, βN the Stone-Čech compactification of N such that $N\subset\beta N$ and the points of $\beta N-N$ are free ultrafilters on N, $N^*=\beta N-N$, P the set of all permutations of N, f^* with f in P the continuous extension of f to a mapping (homeomorphism) of βN onto itself, and P^* the set of all f^* with f in P.

1. **Types and production of types.** Recall that the cardinal of βN is $\exp\exp\aleph_0$, the cardinal of every dense set in N^* is at least $\exp\aleph_0$, and any discrete countable subset of βN is normally embedded in βN.

1.1. Let T be a set and τ be a mapping of $N^*=\beta N-N$ onto T such that $\tau x=\tau y$ if and only if $p^*x=y$ for some p in P. The elements of T are called the types of ultrafilters on N, and if $t=\tau x$ then t is called the type of x and x is said to be of type t. The set $\tau^{-1}[(t)]$ of all $x\in N^*$ of type t is of cardinal $\exp\aleph_0$ because the cardinal of P is $\exp\aleph_0$ and $\tau^{-1}[(t)]$ is clearly dense in N^*. It follows that the cardinal of T is $\exp\exp\aleph_0$.

If M is any countable infinite set and $f\colon M\to N$ is a bijective mapping then the type of any free ultrafilter x on M is defined to be the type of the "image" of x under f. Clearly the definition does not depend on f.

1.2. If X is any collection of ultrafilters on a set M and if y is an ultrafilter on X then the sum of X, with respect to y designated by $\sum_y X$, is defined to be the collection z of all sets of the form $\bigcup\{M_x\mid x\in Y\}$ where $Y\in y$ and $M_x\in x$ for each x in Y. It is easy to show that z is actually an ultrafilter on M. A collection X of ultrafilters is called to be discrete if there exists a disjoint family $\{M_x\mid x\in X\}$ with $M_x\in x$. In a natural way we apply those definitions to a one-to-one family $\{x_a\}$ of ultrafilters and an ultrafilter on the index set.

Now let $\{x_n\}$ and $\{x_n'\}$ be two discrete sequences of ultrafilters on a countable set M such that $\tau x_n=\tau x_n'$ for each n. Then the sums $\sum_y\{x_n\}$ and $\sum_y\{x_n'\}$ are of the same type for any ultrafilter y on N, and so we may introduce the following definition.

1.3. Definition. If $\{t_n\}$ is any sequence of types and y is any ultrafilter on N then the sum t of $\{t_n\}$ with respect to y is designated by $\sum_y\{t_n\}$ and defined to be the type of any $\sum_y\{x_n\}$ with $\{x_n\}$ a discrete sequence of ultrafilters such that $\tau x_n=t_n$. It is clear that then any x of type t is of the form $\sum_y\{x_n\}$.

Now we are prepared to introduce the main concept—the producing relation.

1.4. Definition. The producing relation ϕ on T is defined to be the set of all pairs $\langle u, v\rangle$ such that $v=\sum_y\{t_n\}$ for some y of type u

and some sequence $\{t_n\}$ in T. Thus, the domain of ϕ is T, its range is contained in T and $\phi[(t)] = E\{\sum_y \{t_n\} \mid y \in t\}$. The symbol $\langle u, v\rangle \in \phi$ will often be read either "u produces v" or "v is produced by u." The main result reads as follows.

THEOREM C. *Any type is produced by at most* $\exp \aleph_0$ *types, and any type produces* $\exp \exp \aleph_0$ *types, i.e.* $\operatorname{card} \phi^{-1}[(t)] \leqq \exp \aleph_0$, $\operatorname{card} \phi[(t)] = \exp \exp \aleph_0$ *for any type* t.

The proof will be given in 1.7 and 1.9 below after we develop a topological interpretation of the relation ϕ.

1.5. It is easy to see that a countable $X \subset N^*$ is a discrete subset of the topological space βN if and only if X is discrete in the sense of 1.2, and that $\operatorname{cl} X$ is homeomorphic to βN if X is infinite. Thus given a z in $\operatorname{cl} X - X$, the traces of neighborhoods of z on X form an ultrafilter z_X on X whose type will be denoted by $\tau_X z$ and called the type of z relative to X. Clearly $z = \sum_{z_X} X$ and so $\langle \tau_X z, tz\rangle \in \phi$. If y is any free ultrafilter on X then $z = \sum_y X$ belongs to $\operatorname{cl} X - X$ and $y = z_X$.

Now let $\{t_n\}$ be any sequence of types and t be a type. Choose a discrete sequence $\{x_n\}$ of representatives (that means $\tau x_n = t_n$), and consider the set of all x_n. The set of all $\sum_y \{t_n\}$, $\tau y = t$, coincides with the set $E\{\tau z \mid z \in \operatorname{cl} X - X, \tau_X z = t\}$. So $\phi[(t)] = E\{\tau z \mid z \in \operatorname{cl} X - X, \tau_X z = t$ for some discrete countable $X \subset N^*\}$. In what follows we shall use that topological interpretation without any reference.

1.6. Let $\{M_n\}$ be a countable decomposition of N and let x_n, $y_n \in \operatorname{cl} M_n - M_n$. If $x_n \neq y_n$ for each n then $\operatorname{cl} E\{x_n\} \cap \operatorname{cl} E\{y_n\} = \emptyset$ and conversely. The proof is evident.

1.7. The second statement of Theorem C follows immediately from 6. Indeed taking any decomposition $\{M_n\}$ of N with all M_n infinite, we have $\operatorname{card} \operatorname{cl} M_n - M_n = \exp \exp \aleph_0$ for each n and therefore we get $\exp \exp \aleph_0$ disjoints sets $\operatorname{cl} X = \operatorname{cl} E\{x_n\}$ each containing at least one point y with $\tau_X y = t$, which is of type in $\phi[(t)]$. Since $\operatorname{card} \tau^{-1}[(t)] = \exp \aleph_0$, the result follows.

To prove the first statement of Theorem C we need the following lemma.

1.8. Let $y \in \beta N - N$. There exists a set $\mathfrak{X}$, $\operatorname{card} \mathfrak{X} \leqq \exp \aleph_0$, of discrete countable subsets X of $\beta N - N$ such that if Y is any discrete countable subset of $\beta N - N$, and if $y \in \operatorname{cl} Y - Y$, then $Y \supset X$ for some $X \in \mathfrak{X}$.

PROOF. For each countable decomposition $\{M_n\}$ of N choose an $x_n \in \operatorname{cl} M_n - M_n$ such that $y \in \operatorname{cl} X - X$, if possible, and take all $X' \subset X$ with $y \in \operatorname{cl} X' - X$. The set $\mathfrak{X}$ of all X', $\{M_n\}$ variable, has required properties by 1.6. The cardinal of $\mathfrak{X}$ is at most $\exp \aleph_0 \exp \aleph_0 = \exp \aleph_0$.

1.9. To prove the first statement of Theorem C it now suffices to combine 1.7 with 1.5 and the following simple observation: If $X \subset Y$, $y \in \text{cl } X - Y$ and Y is a discrete subset of $\beta N - N$, then $\tau_X y = \tau_Y y$, i.e. τ_X is a restriction of τ_Y.

THEOREM C′. *Let* $\phi_\infty = \bigcup \{\phi^k \mid k \in N,\ k \neq 0\}$, $\phi_\infty^{-1} = \bigcup \{(\phi^{-1})^k \mid k \in N,\ k \neq 0\}$, *where* $\rho^k = \rho \circ \cdots \circ \rho$ *(k-times). Then* $\phi_\infty^{-1} = (\phi_\infty)^{-1}$, *and* card $\phi_\infty[(t)] = \exp \exp \aleph_0$, card $\phi_\infty^{-1}[(t)] \leq \exp \aleph_0$ *for any t in T.*

2. **Applications.** A space is called homogeneous if any point can be mapped onto any point by an autohomeomorphism. W. Rudin proved in [5] that the space N^* is not homogeneous by proving the existence of the so called P-points. His proof of the existence of P-points heavily depends on the continuum hypothesis. Theorem C enables us to prove the nonhomogeneity of N^* without the continuum hypothesis.

2.1. Proof of nonhomogeneity of N^*. For each x in N^* denoted by T_x the set of all relative types of x; i.e. $T_x = \phi^{-1}[\tau x]$. If $lx = y$ for some autohomeomorphism l of N^*, then clearly $T_x = T_y$. Since the sets T_x are of the cardinals at most $\exp \aleph_0$, card $T = \exp \exp \aleph_0$ and $\{T_x \mid x \in N^*\}$ is a covering of T, the result follows.

REMARK. It should be remarked that we have proved the existence of $\exp \exp \aleph_0$ equivalence classes. Those equivalence classes define "free types" of free ultrafilters. Relative free types are defined similarly.

It remains to prove Theorems A′ and B′.

2.2. LEMMA. *There exists a disjoint transfinite family* $\{T_\alpha \mid \alpha < \omega_1\}$ *of subsets of T and a family* $\{t_\alpha \mid \alpha < \omega_1\}$, $t_\alpha \in T$, *such that, denoting by* X_α *the set of all points of* βN *of types in* T_α, *each countable discrete subset X of* $\bigcup \{X_\beta \mid \beta < \alpha\}$ *has a cluster point in* X_α *of type* t_α *with respect to X.*

THEOREM D. *For any set A of countable ordinals let* $P_A = N \cup \bigcup \{X_\alpha \mid \alpha \in A\}$ *be a subspace of* βN. *If* $\{A_b \mid b \in B\}$ *is a countable family of sets of countable ordinals, then the product* $P = \prod \{P_{A_b} \mid b \in B\}$ *is countably compact if* $\bigcap \{A_\beta\}$ *is unbounded, and it is not countably compact if that intersection is empty.*

First we prove Theorem D, then Theorems A′, B′, and finally the main step, the Lemma.

2.3. PROOF OF THEOREM D. If the intersection is empty then the "diagonal" is a closed infinite discrete subspace of the product. For the converse, assume that the intersection is an unbounded set A, and let $\{z(n)\}$ be a sequence in P. Denoting by π_b the projection from P onto P_{A_b} we can choose a subsequence $\{y(n)\}$ such that each sequence $\{\pi_b y(n)\}$ is either eventually constant or eventually one-to-one. Choose an $\alpha \in A$ so that each $\pi_b y(n)$ belongs to $\bigcup\{X_\beta \mid \beta < \alpha\}$. Choose any point $y \in \beta N$ of type t_α and consider the point $z = \{z_b\}$ of P defined as follows: if $\{\pi_b y(k)\}$ is eventually constant, then z_b is this constant; otherwise z_b is the image of y under the mapping $\{n \to \pi_b y(n)\}: N \to N$. It can be proved that z is a cluster point of $\{y(n)\}$, see [3; the proof of E], and so of $\{z(n)\}$.

2.4. Proof of Theorem A′. For $0 \leqq k \leqq n$ let A_k be the class of countable ordinals which are not congruent to k modulo $n+1$. Of course $\bigcap\{A_k \mid k \leqq n\} = \emptyset$ and the intersection of any proper subfamily of $\{A_k \mid k \leqq n\}$ is unbounded.

2.5. Proof of Theorem B′ is similar. For each $k \in N$ let A_k be the set of all ordinals which are not congruent to k modulo ω_0.

It remains to prove Theorem D. The following simple consequence of Theorem C′ will be needed.

2.6. If $T' \subset T$ is of cardinal at most $\exp \aleph_0$ and if $T_1 \subset T$ is of cardinal $\exp \exp \aleph_0$, then $T' \cap \phi^\infty[(t)] = \emptyset$ for $\exp \exp \aleph_0$ of $t \in T_1$. Indeed, by Theorem C′ each set $\phi_\infty^{-1}[(t)]$ is of cardinal at most $\exp \aleph_0$.

2.7. Proof of Lemma. We shall prove the existence of $\{T_\alpha\}$ and $\{t_\alpha\}$ with the following additional properties:

(a) card $T_\alpha \leqq \exp \aleph_0$;

(b) T_α, $\alpha > 0$, consists precisely of the types of points of βN whose types with respect to some discrete subset of $\bigcup\{X_\beta \mid \beta < \alpha\}$ is t_α.

(c) $\phi[(t_\alpha)] \cap (\bigcup\{T_\beta \mid \beta < \alpha\}) = \emptyset$.

Starting with any $t = t_0$, $T_0 = (t_0)$ the induction goes by 2.6.

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