THE GROTHENDIECK GROUP FOR STABLE HOMOTOPY IS FREE

BY PETER FREYD[1]

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Let H_n^m be the set of homotopy types of base-pointed finite complexes of dimension $\leqq m$ and connectivity $\geqq n$. We shall always assume that $2n \geqq m$, in other words, that we are working in the "stable range".

H_n^m is closed under the "wedge" operation ($X \vee Y$ is obtained by identifying the base points in the disjoint union of X and Y). Chang [1] has classified the wedge indecomposables in the case $m \leqq n+3$ and has shown that a unique wedge decomposition theorem holds in H_n^{n+3}, $n \geqq 3$.

PROPOSITION 1. *Unique wedge decomposition fails in* H_5^{10}. *Indeed* $(H_5^{10}, \vee)$ *fails to be a cancellation semigroup. The same pathology holds for any* H_n^m, $m \geqq n+5$, $2n \geqq m$.

The easiest example: Let $\nu \in \pi_9(S^6)$ be a map of order 8. Let Cone(ν) be its mapping cone. Then $S^6 \vee \text{Cone}(\nu) \simeq S^6 \vee \text{Cone}(3\nu)$ but Cone(ν) $\not\simeq$ Cone(3ν). (The isomorphism uses only that 3 is prime to the order of ν, the nonisomorphism uses only that 3 is not congruent to ± 1 mod the order of ν. ν could not be of order 2, 3, 4, or 6. Hence a similar example is avoided in the range covered by Chang.)

Let C_n^m be the cancellation semigroup obtained from $(H_n^m, \vee)$ by defining $X \equiv Y$ if there exists Z such that $X \vee Z \simeq Y \vee Z$.

THEOREM 2. $X \equiv Y$ *iff for the bouquet of spheres, B, with the same Betti numbers as X it is the case that* $X \vee B \simeq Y \vee B$.

It follows that the inclusion $H_n^m \to H_n^{m+1}$ remains a monomorphism when we pass to $C_n^m \to C_n^{m+1}$. The suspension functor preserves wedges and hence we obtain a homomorphism from $(H_n^m, \vee)$ to $(H_{n+1}^{m+1}, \vee)$. By Freudenthal's theorem $H_n^m \to H_{n+1}^{m+1}$ is an isomorphism. We obtain a family of monomorphisms $C_n^m \to C_{n'}^{m'}$, $n \leqq n'$, $m \leqq m'$ the direct limit of which we'll call S. Each C_n^m is a sub-semigroup of S and it may be noted that each of the statements below about S and its ambient group specializes nicely to C_n^m and its ambient group.

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COROLLARY 3. *Given* $\gamma \in \pi_n(S^m)$, $n \neq m$, $\gamma \neq 0$ *then* $[Cone(\gamma)]$ *is indecomposable in* S.

PROPOSITION 4. *S is not free, i.e. it is not a unique factorization semigroup.*

The easiest example. Let $\alpha \in \pi_9(S^6)$ be a map of order 3, $\nu \in \pi_9(S^6)$ of order 8. Then $\text{Cone}(\alpha) \vee \text{Cone}(\nu) \simeq S^6 \vee S^{10} \vee \text{Cone}(\alpha+\nu)$. The example lives in H_5^{10}. For any γ, $\delta \in \pi_n(S^m)$, $n \neq m$, γ, δ of co-prime orders the same pathology may be exhibited.

The above example depends upon the mixing of prime integers. We may explicate that dependency by defining a space X to be p-**primary** (p a prime integer) if there exist maps $f: B \to X$, $g: X \to B$, where B is the bouquet of spheres with the same Betti numbers as X, such that $gf = p^n \cdot 1_B$, some n.

Spheres are p-primary for any p. The only spaces which are p-primary for more than one p are bouquets of spheres.

Let S_p be the cancellation semigroup obtained by restricting attention to p-primary spaces.

THEOREM 5. *S_p is free, i.e. a unique factorization semigroup. Moreover $[X]$ is indecomposable in S_p iff X is wedge indecomposable.*

(It is known that a space in the stable range is wedge indecomposable iff its only idempotent endomorphisms are 0 and 1 [2].)

Let G be S made into a group (the Grothendieck group for stable homotopy). Let $B: G \to G$ be the map which sends $[X]$ to the bouquet of spheres $[B_X]$ with the same Betti numbers as X. B is idempotent. Let G_S be its image, G^* its kernel. $G = G_S \oplus G^*$. Note that G_S is clearly freely generated by the spheres. Let G_p^* be the subgroup of G^* generated by elements of the form $[X] - [B_X]$ where X is p-primary. Note that S_p made into a group is $G_S \oplus G_p^*$. Hence G_p^* is free.

THEOREM 6. *G^* is the internal direct sum of the G_p^*'s. G is free. It is freely generated by the set* $\{S^n \mid S^n \text{ an } n\text{-sphere}\} \cup \{[X] - [B_x] \mid X \text{ a wedge indecomposable primary space}\}$.

The next was a contention of Milnor.

THEOREM 7. $[X] - [Y]$ *has zero component in* $G_S \oplus G_p^*$ *iff X and Y have the same Betti numbers and there exists* $f: X \to Y$ *such that* $H_*(f; Z_p)$ *is an isomorphism where* Z_p *can be interpreted either as the prime field or the p-adic integers.*

COROLLARY 8. *With the smash product as multiplication,* G_p^* *is an ideal.*

The proofs rely heavily upon the representation of the stable homotopy category $\mathcal{S}$ (of which G is the Grothendieck group) as the full subcategory of projectives in a Frobenius category $\mathfrak{F}$ [3]. (A Frobenius category is an Abelian category in which projectives and injectives coincide and in which there are enough of them in both senses.) Statements 1 through 4 require repeated use of the Schanuel lemma applied in $\mathfrak{F}$. Theorem 5 depends upon a suitable modification of the Nakayama lemma. Theorem 6 uses the Schanuel lemma to represent the Grothendieck group in another more easily handled Grothendieck group arising from $\mathfrak{F}$. For Theorem 7 it is necessary to localize $\mathfrak{F}$ by factoring out, a la Gabriel, the Serre class of objects whose identity maps are of finite order prime to p. This p-localization of $\mathfrak{F}$ has many nice properties: it is Frobenius; its indecomposable injectives are spaces and they are absolutely indecomposable, i.e. are essential extensions of every nontrivial subobject; each of its objects has an injective envelope; it is self-dual and hence each of its objects has a projective co-envelope.

We obtain an almost-answer to the question which inaugurated the investigation: can mapping cones in the stable range be identified by their homotopy properties?

THEOREM 9. *If*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow SX \xrightarrow{Sf} SY$$

is such that an exact sequence of abelian groups results whenever a co-representable functor is applied (or if preferred, whenever any cohomology theory is applied) then $[Z]$ *is equal to* $[\text{Cone}(f)]$ *in the Grothendieck group, i.e.* $Z \equiv \text{Cone}(f)$.

There does exist a sequence

$$S^9 \xrightarrow{\nu} S^6 \longrightarrow \text{Cone }(3\nu) \longrightarrow S^{10} \xrightarrow{S\nu} S^7$$

satisfying the hypothesis. Hence Z need not be of the same homotopy type as Cone(f).

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