PRODUCING PL HOMEOMORPHISMS BY SURGERY

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In the present note we apply the surgery techniques of [4] to the problem of deforming a homotopy equivalence $f: (M, \partial M) \rightarrow (X, \partial X)$ between two p.l. (piecewise linear) manifolds until it is a p.l. homeomorphism. First of all, note that if f can be so deformed, the induced cohomology homomorphism $f^*: H^*(X; Q) \rightarrow H^*(M; Q)$ must pull back the rational Pontryagin classes, which are combinatorial invariants; that is, we must have $f^*(p_i(X; Q)) = p_i(M; Q)$ (cf. [1, Chapter XVI]). For the fixed p.l. manifold $(X, \partial X)$ let $\mathrm{PL}^p(X, \partial X)$ denote the "h-cobordism classes" of such homotopy equivalences. Then the problem of deforming f to a p.l. homeomorphism is a special case of the problem of determining $\mathrm{PL}^p(X, \partial X)$. Because if $\mathrm{PL}^p(X, \partial X)$ can be shown to have only one element, then f is h-cobordant to $id: (X, \partial X) \rightarrow (X, \partial X)$ and hence homotopic to a p.l. homeomorphism by the p.l. h-cobordism theorem (cf. [5]). The main result (Theorem 1.4) computes a finite upper bound for the number of elements in $\mathrm{PL}^p(X, \partial X)$. Finally, the topological invariance of the rational Pontryagin classes recently demonstrated by S. P. Novikov [3] implies that any topological homeomorphism $f: (M, \partial M) \rightarrow (X, \partial X)$ satisfies $f^*(p_i(X; Q)) = p_i(M; Q)$. Thus the Hauptvermutung holds for those p.l. manifolds $(X, \partial X)$ with ord $\mathrm{PL}^p(X, \partial X) = 1$ (cf. Corollary 1.7). Recall that the Hauptvermutung claims that topologically homeomorphic p.l. manifolds are piecewise linearly homeomorphic. Unless otherwise stated we shall work entirely within the p.l. category in what follows.

Let $(X, \partial X)$ be a finite CW pair satisfying Poincaré duality in dimension n. A *piecewise linear structure on the homotopy type of* $(X, \partial X)$, i.e., a *p.l. structure* on $(X, \partial X)$, is a piecewise linear n-manifold $(M, \partial M)$ together with a homotopy equivalence $f: (M, \partial M) \rightarrow (X, \partial X)$. Two such p.l. structures $f_i: (M_i, \partial M_i) \rightarrow (X, \partial X)$ $(i = 0, 1)$ are *equivalent* provided they are h-cobordant: that is, there is an h-cobordism $(W, \partial_c W)$ between the $(M_i, \partial M_i)$ so that $\partial W = M_0 \cup \partial_c W \cup M_1$ and W (resp. $\partial_c W$) is an h-cobordism between the M_i (resp. ∂M_i); and there is a homotopy equivalence

$$\bar{f}: (W; M_0, \partial_c W, M_1) \rightarrow (X \times [0, 1]; X \times 0, \partial X \times [0, 1], X \times 1)$$
$$\text{for which } \bar{f} \mid M_i = f_i \ (i = 0, 1).$$

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The resulting set of equivalence classes of p.l. structures on $(X, \partial X)$ will be denoted by $\mathrm{PL}(X, \partial X)$. When $\partial X=0$, set $\mathrm{PL}(X, \partial X)=\mathrm{PL}(X)$. If $(X, \partial X)$ is actually a p.l. manifold, then a p.l. structure $f\colon (M, \partial M) \to (X, \partial X)$ is a *Pontryagin p.l. structure* on $(X, \partial X)$ provided $f^*(p_i(X; Q))=p_i(M: Q)$. If $f^*(\tau X\oplus\epsilon^1)=\tau M\oplus\epsilon^1$, where τX and τM are the tangent microbundles of X and M (cf. [2]), then f is said to be a *tangential p.l. structure* on $(X, \partial X)$. Let $\mathrm{PL}^p(X, \partial X)$ and $\mathrm{PL}^\tau(X, \partial X)$ denote respectively the equivalence classes of Pontryagin and tangential p.l. structures on $(X, \partial X)$.

D. Sullivan in [6] has independently studied p.l. structures on a homotopy type and has very comprehensive results which generalize ours. His work is roughly the analogue in the homotopy category of the Lashof-Rothenberg smoothing theory of [7].

The content of the next two theorems is that in trying to deform a homotopy equivalence to a p.l. homeomorphism or in computing $\mathrm{PL}(X, \partial X)$ we can simplify matters by passing to the stable range. Let ξ denote an oriented k-disc bundle $\pi\colon E_x\to X$ over X with total space E_x so that $\dot{E}_x\subset E_x$ denotes the total space of the associated sphere bundle. Let $(WE_x, \partial WE_x)=(E_x, \dot{E}_x\cup E_x|\partial X)$. This is a Poincaré pair in dimension $n+k$. If $f\colon (M, \partial M)\to(X, \partial X)$ is a p.l. structure, then the total space $E_m=f^*E_x$ is an $(n+k)$-manifold with boundary $(WE_m, \partial WE_m)=(E_m, \dot{E}_m\cup E_m|\partial X)$ and the natural bundle map $\bar{f}\colon E_m \to E_x$ gives rise to a p.l. structure $\xi(f)\colon (WE_m, \partial WE_m) \to(WE_x, \partial WE_x)$. This procedure defines a map

$$\xi\colon \mathrm{PL}\ (X, \partial X) \to \mathrm{PL}\ (WE_x, \partial WE_x).$$

Furthermore, whenever X is a p.l. manifold the microbundle equation $\tau E_x=\pi^*(\tau X\oplus E_x)$ (cf. [2, Theorem 5.9]) implies that ξ takes Pontryagin and tangential structures into Pontryagin and tangential structures respectively.

Similar definitions may be made in the smooth category and 2.1, 1.2, and 1.3 below remain valid when transferred to the smooth realm.

THEOREM 1.1. *Suppose* $n\geqq 6$, $k\geqq 3$ (*unless* $n=6$ *or* 14 *in which case* $k>n/2+2$) *and* X *and* $\partial X\neq 0$ *are* 1-*connected. Then* $\xi\colon \mathrm{PL}(X, \partial X) \to\mathrm{PL}(WE_x, \partial WE_x)$ *is a* 1-1 *correspondence. Similarly for Pontryagin and tangential p.l. structures.*

Interesting examples are $E_x=$ trivial k-bundle and $E_x=$ stable normal bundle of X (whenever X is a manifold).

Now suppose $(X, \partial X)$ is actually an n-submanifold with a normal bundle of an $(n+k)$-manifold $(W_x, \partial W_x)$. Let $(W_m, \partial W_m)$ be another

$(n+k)$-manifold and $f\colon (W_m, \partial W_m)\to(W_x, \partial W_x)$ be a homotopy equivalence that is h-regular (cf. [4]) on $(X, \partial X)$ with $(M, \partial M) = f^{-1}(X, \partial X)$.

THEOREM 1.2. *Suppose X, $\partial X\neq 0$, W_x, and ∂W_x are simply connected, and let $n\geqq 6$ and $k\geqq 3$ (unless $n=6$ or 14 in which case $k>n/2+2$). If $f\colon (W_m, \partial W_m)\to(W_x, \partial W_x)$ is homotopic to a p.l. homeomorphism, then so is $f\colon (M, \partial M)\to(X, \partial X)$.*

COROLLARY 1.3. *Suppose $f\colon (M, \partial M)\to(X, \partial X)$ is a homotopy equivalence of n-manifolds covered by a bundle map $\bar f\colon E_m\to E_x$, where E_x and E_m are oriented k-disc bundles over X and M. Suppose X and $\partial X\neq 0$ are 1-connected, and n and k satisfy the conditions of Theorem 1.2. Then f is homotopic to a p.l. homeomorphism iff the same is true of $\bar f\colon (WE_m, \partial WE_m)\to(WE_x, \partial WE_x)$.*

For example $f\colon (M, \partial M)\to(X, \partial X)$ is homotopic to a p.l. homeomorphism iff $f\times id\colon (M, \partial M)\times(D^k, \partial D^k)\to(X, \partial X)\times(D^k, \partial D^k)$ is for some large k.

REMARK 1. Theorem 1.1 follows from 1.8 of [4], whereas the proof of 1.2 requires 1.8 of [4] plus the h-cobordism theorem. These results may be relativized by considering n-dimensional Poincaré triples $(X; \partial_1 X, \partial_2 X)$ where $\partial_2 X$ is actually an $(n-1)$-manifold and homotopy equivalences $f\colon (M; \partial_1 M, \partial_2 M)\to(X; \partial_1 X, \partial_2 X)$ where ∂M breaks up as the union $\partial_1 M\cup\partial_2 M$ of two submanifolds having $\partial(\partial_1 M)=\partial(\partial_2 M)=\partial_1 M\cap\partial_2 M$ and $f\colon \partial_2 M\to\partial_2 X$ is a p.l. homeomorphism. We then have "p.l. structures on X rel $\partial_2 X$" and denote the h-cobordism classes of such things by $\mathrm{PL}(X \text{ rel } \partial_2 X)$. In this relative setting X and $\partial_1 X\neq 0$ should be 1-connected.

The main results on $\mathrm{PL}^p(X, \partial X)$ are:

THEOREM 1.4. *Let $(X, \partial X)$ be a p.l. manifold of dimension at least six with X and $\partial X\neq 0$ simply connected. Then $\mathrm{PL}^p(X, \partial X)$ is finite and*

$$\text{ord } \mathrm{PL}^p(X, \partial X) \leqq \text{ord } (\text{Torsion } H^{4*}(X; Z))\cdot \text{ord } H^{4*+2}(X; Z_2).$$

Assume for 1.5, 1.6, 1.7 that $(X, \partial X)$ is as in Theorem 1.4.

COROLLARY 1.5. *Suppose that $f\colon (M, \partial M)\to(X, \partial X)$ is a homotopy equivalence between p.l. manifolds, and that*

(i) $f^*(p_i(X; Q))=p_i(M; Q)$,

(ii) Torsion $H^{4*}(X; Z)=0=H^{4*+2}(X; Z_2)$.

Then f is homotopic as a map of pairs to a p.l. homeomorphism.

Recall that by [3] any topological homeomorphism satisfies (i) of 1.5. Thus

COROLLARY 1.6. *The number of piecewise linearly distinct p.l. manifolds which are topologically equivalent to X is finite and is less than or equal to*

$$\text{ord (Torsion } H^{4*}(X; Z)) \cdot \text{ord } H^{4*+2}(X; Z_2).$$

COROLLARY 1.7. *The Hauptvermutung is true for X provided it satisfies* (ii) *of* 1.5.

REMARK. Theorem 1.1 of [4] together with 1.4 imply that 1.4 through 1.7 are also valid whenever X is a closed simply connected manifold of dimension at least six. In this case X-point replaces X in the cohomology conditions of 1.5 and in the upper bound formulas.

The proof of the main result involves an analysis of the problem of deforming a homotopy equivalence $f: (M, \partial M) \to (X, \partial X)$ to a p.l. homeomorphism. The major steps in this approach are

(a) Choose a handlebody decomposition $X = X_0 \cup \cdots \cup X_n$ of X, where X_i = union of the handles of index i, and find a decomposition of M as a union of submanifolds $M_0 \cup \cdots \cup M_n$ with $M_i \cap M_j = \partial M_i \cap \partial M_j$ so that f may be deformed until it is a homotopy equivalence of M_i onto X_i.

(b) Deform f until it is a p.l. homeomorphism of M_0 to X_0, and then assuming that f is a p.l. homeomorphism of $M_0 \cup \cdots \cup M_i$ onto $X_0 \cup \cdots \cup X_i$, try to deform it rel $M_0 \cup \cdots \cup M_{i-1}$ to produce a p.l. equivalence of $M_0 \cup \cdots \cup M_{i+1}$ onto $X_0 \cup \cdots \cup X_{i+1}$.

Step (a) can always be accomplished using Theorem 1.1 of [4], but there is in general an obstruction in Step (b). Consider a single k-handle $(X_k; \partial_- X_k, \partial_+ X_k)$ of the form $X_k = \partial_- X_k \times [0, 1] \cup_h D^k \times D^{n-k}$, where $h: S^{k-1} \times D^{n-k} \to \partial_- X_k \times 1$ is an imbedding, $\partial_- X_k = \partial_- X_k \times 0$, and $\partial_+ X_k = (\partial_- X_k \times 1 - \text{int } h(S^{k-1} \times D^{n-k})) \cup D^k \times S^{n-k-1}$. Let $D^k \subset X_k$ denote the *core disc* $= h(S^{k-1} \times 0) \times [0, 1] \cup D^k \times 0$. Let M be an n-manifold with $\partial M = \partial_- M \cup \partial_+ M$ and $f: (M; \partial_- M, \partial_+ M) \to (X_k; \partial_- X_k, \partial_+ X_k)$ be a homotopy equivalence which is a p.l. homeomorphism of $\partial_- M$ onto $\partial_- X_k$. Then we have

PROPOSITION 1.8. *If $k \geqq 5$, $n - k \geqq 3$, and $\partial_- X_k$ is simply connected, then f (as a map of triples) may be deformed* rel $\partial_- M$ *to a p.l. homeomorphism iff $c^k(f) = 0$, where $c^k(f)$ is the obstruction of Theorem* 1.2 *of* [4] *to deforming f* rel ∂M *until it is h-regular on the core disc.*

Recall that $c^k(f) = 0$ if k is odd.

Consider next a two-stage handlebody $(X; \partial_- X, \partial_+ X) = (X_k \cup X_{k+1}; \partial_- X_k, \partial_+ X_{k+1})$ built up from $\partial_- X$ by first adding a k-handle X_k and then a $(k+1)$-handle X_{k+1} in such a way that $H_k(X, \partial_- X) = Z_l (l > 0)$. Suppose M is an n-manifold which decomposes as a union $M_k \cup M_{k+1}$ of n-submanifolds with $M_k \cap M_{k+1} = \partial M_k \cap \partial M_{k+1}$, and suppose $\partial M = \partial_- M \cup \partial_+ M$. Let

$$f\colon (M_k \cup M_{k+1}; \partial_- M, \partial_+ M) \to (X_k \cup X_{k+1}; \partial_- X, \partial_+ X)$$

be a homotopy equivalence respecting decompositions which is a p.l. homeomorphism of $\partial_- M$ onto $\partial_- X$.

PROPOSITION 1.9. *Suppose $\partial_- X$ is 1-connected, $k \geqq 5$, and $n-k \geqq 4$ (unless $k=6$ or 14 in which case $n-k > k/2+1$). Then*

(i) $l \cdot c^k(f) = 0$.

(ii) *If f is already a p.l. homeomorphism of M_k onto X_k and f': $M_k \cup M_{k+1} \to X_k \cup X_{k+1}$ is another such homotopy equivalence homotopic to f rel ∂M, then $c^{k+1}(f') - c^{k+1}(f)$ is divisible by l. In fact, when $k+1 \equiv 0$ or 2 mod 4, for any integer m there is such an f' with $c^{k+1}(f') - c^{k+1}(f) = m \cdot l (m \cdot l \in Z_2$ if $k+1 \equiv 2$ mod 4).*

Finally, let $(X, \partial X)$ be an n-manifold obtained from $(X', \partial X')$ by adding a $4s$-handle X_{4s} to the 1-connected $\partial X'$. Suppose M is an n-manifold which decomposes as a union of n-submanifolds $M' \cup M_{4s}$, and $f\colon (M, \partial M) \to (X, \partial X)$ is a homotopy equivalence that respects the decompositions and is a p.l. homeomorphism of M' onto X'.

PROPOSITION 1.10. *If the transverse disc $(0 \times D^{n-4s}, 0 \times S^{n-4s-1}) \subset (X, \partial X)$ represents an indivisible integral cohomology class in $H^{n-4s}(X, \partial X)$ and $f^*(p_i(X; Q)) = p_i(M; Q)$ for all $i \leqq 4s$, then $c^{4s}(f) = 0$ provided $4s \geqq 8$ and $n - 4s \geqq 3$.*

The proof of 1.4 proceeds by induction up over successive skeletons X_k in a canonical handlebody decomposition of X using 1.8 through 1.10 and

LEMMA 1.11. *Suppose X is a 1-connected p.l. manifold of dimension at least six with ∂X also 1-connected. Then two p.l. structures $f\colon (M_i, \partial M_i) \to (X^n, \partial X^n)$ $(i=0, 1)$ are h-cobordant iff $\bar{f}_1 f_0\colon (M_0, \partial M_0) \to (M_1, \partial M_1)$ is homotopic to a p.l. homeomorphism, where $\bar{f}_1$ is any homotopy inverse to f_1.*

To circumvent difficulties arising in the proof 1.4 when $n-k$ is small, we replace X by $X \times D^l$ in virtue of 1.1. Also, special arguments must be given for the low dimensional induction steps when $k = 2$, 3, or 4.

Corresponding to the existence problem of trying to deform a homotopy equivalence to a p.l. homeomorphism there is the uniqueness question of when two homotopic p.l. homeomorphisms are pseudo-isotopic. Theorem 1.12 below is a representative result, the proof of which consists roughly in crossing the existence methods by the circle. If f is a p.l. homeomorphism of an n-manifold $(X, \partial X)$ onto itself, form the *mapping torus* $(X_f, \partial X_f)$ from $X \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$. A homotopy $f_t: (X, \partial X) \to (X, \partial X)$ of $f_0 = \text{identity}$ and $f_1 = f$ gives rise to a map $f_t: (X_f, \partial X_f) \to (X \times S^1, \partial X \times S^1)$.

THEOREM 1.12. *Let f be a p.l. homeomorphism of $(X^n, \partial X^n)$ onto itself which is homotopic to the identity as a map of pairs via f_t. Suppose X^n and ∂X^n are 1-connected and $n \geq 6$. Assume further that*

(a) $f_t^*(p_i(X \times S^1; Q)) = p_i(X_f; Q)$, *and*

(b) *Torsion* $H^{4*-1}(X; Z) = 0 = H^{4*+1}(X; Z_2)$.

Then f_t is homotopic rel $X \times 0 \cup X \times 1$ *to a p.l. pseudo-isotopy between $f_0 = \text{identity}$ and $f_1 = f$. The deformation of f_t takes $\partial X \times [0, 1]$ into $\partial X \times [0, 1]$.*

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