A GENERAL WEDDERBURN THEOREM

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Let R be a ring, E a left R-module, and set

$$R'(E) = R' = \operatorname{End}_R E, \qquad R''(E) = R'' = \operatorname{End}_{R'} E.$$

We say that E is *balanced* in case the natural homomorphism $\lambda: R \to R''$ under which $x \mapsto \lambda_x$ where $\lambda_x(v) = xv$, $\forall v \in E$, is an isomorphism. The classical Wedderburn theorem gives a criterion for a module to be balanced. We give a very short proof of a theorem of Morita (in the terminology of [1]) which implies many such criteria.

A left R-module E is said to be a *generator* (for left R-modules) if every R-module can be expressed as a homomorphic image of (possibly infinite) direct sum of copies of E.

THEOREM 1. *Let E be a generator. Then E is balanced.*

PROOF. We first prove that for any module F, $R \oplus F$ is balanced. Given $v \in R \oplus F$, there exists an element $\phi \in R'(R \oplus F)$ such that $\phi(1) = v$ (we view R and F as embedded in $R \oplus F$ as $R \oplus 0$ and $0 \oplus F$ respectively). Let $p: R \oplus F \to R$ be the projection. Let $f \in R''(R \oplus F)$. Then $f(1) = fp(1) = pf(1)$. Hence $f(1) \in R$. It follows that

$$f(v) = f\phi(1) = \phi f(1) = \phi(f(1) \cdot 1) = f(1)\phi(1) = f(1)v.$$

This proves what we wanted.

Let E be a generator. There exists a surjective homomorphism $E^n \to R$ for some integer $n \geqq 1$ (we can take n finite because R is generated by one element). Since R is in fact free, we can write $E^n = R \oplus F$ for some module F. Hence E^n is balanced. We conclude the proof with the following lemma.

LEMMA. *If E is any module and E^n is balanced, then E is balanced.*

PROOF. An element $\phi \in \operatorname{End}_R(E^n)$ can be represented by a matrix (ϕ_{ij}) with $\phi_{ij} \in \operatorname{End}_R(E)$, namely for $v \in E^n$ with components $v_j \in E$ we have

$$\phi(v) = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \phi_{n1} & \cdots & \phi_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}.$$

Let $f \in R''(E)$. Then the matrix

$$\begin{bmatrix} f & \cdot & \cdot & \cdot & 0 \\ \cdot & f & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & f \end{bmatrix}$$

operating on E^n commutes with every $\phi=(\phi_{ij})\in R'(E^n)$, and hence there exists an element $x\in R$ such that $(f)=(\lambda_x)$. This proves our lemma.

As applications of Theorem 1, we have the following special cases.

EXAMPLE 1. Let R be a ring without two-sided ideals except 0 and R. Then any left ideal $L\neq 0$ is a generator, because $LR=R$ and hence there exists elements $a_i\in R$ $(i=1,\cdots,n)$ such that $R=\sum_1^n La_i$, whence a surjective homomorphism $L^n\to R$. In this way we recover Rieffel's theorem [6].

EXAMPLE 2. Let R be a Dedekind ring, and I any fractional ideal $\neq 0$. Then I is a generator, again because $II^{-1}=R$, so there exist elements a_i in the quotient field of R such that $R=\sum Ia_i$.

We next prove a converse for Theorem 1 (Morita [5]).

THEOREM 2. *A module E is a generator if and only if E is balanced as an R-module, and finitely generated projective as an R'-module.*

PROOF. Let E be a generator. We have already proved that E is balanced, and we have a left R-module isomorphism $E^n\simeq R\oplus F$ for some module F. Applying $\mathrm{Hom}_R(\cdot, E)$ to E^n, we find the natural additive group isomorphisms

$$R'^n = \mathrm{Hom}_R(E, E)^n \simeq \mathrm{Hom}_R(E^n, E) \simeq \mathrm{Hom}_R(R, E) \oplus \mathrm{Hom}_R(F, E). \tag{1}$$

Each one of the abelian groups appearing in (1) is in fact a left R'-module, the operation of R' on each group on the left being defined as composition of mappings. (This is trivially verified.) Furthermore the isomorphisms in (1) are compatible with this operation. Therefore (1) is a statement of left R'-isomorphism and direct sum. Furthermore $\mathrm{Hom}_R(R, E)$ is R'-isomorphic to E (under the map $f\mapsto f(1)$). Hence E is R'-finitely generated (being an R'-homomorphic image of R'^n) and R'-projective by (1). Conversely, if $R'^n\approx E\oplus F$ for some R'-module F, and $R\simeq\mathrm{Hom}_{R'}(E, E)$, then

$$\mathrm{Hom}_{R'}(R'^n, E) \simeq [\mathrm{Hom}_{R'}(R', E)]^n = E^n$$

yields an isomorphism

$$E^n \simeq R \oplus G,$$

of left R'-modules, where $G = \mathrm{Hom}_{R'}(F, E)$. This proves that E is a generator.

A *simple ring* is a ring (with identity) without two-sided nontrivial ideals.

THEOREM 3. *If R is a simple ring, and if I is a left ideal $\neq 0$, then I is finitely generated projective over $K = \mathrm{End}_R\, I$, and $R \simeq_{\mathrm{nat}} \mathrm{End}_K\, I$. Furthermore, K is a simple ring if and only if I is a finitely generated projective right ideal, and in this case there is a category isomorphism ${}_R\mathfrak{M} \simeq {}_K\mathfrak{M}$.*

PROOF. The first sentence follows from Example 1 and Theorem 2. Also, if I_R is finitely generated projective, then K is known to be simple along with R. Conversely, I is homomorphic to a left ideal I' of K (under right multiplication by elements of I). If K is simple, then ${}_KI'$, whence ${}_KI$, is a generator. Since $R \simeq \mathrm{End}_K\, I$, I_R is finitely generated projective by Theorem 2. Since I is then a progenerator, ${}_K\mathfrak{M} \simeq {}_R\mathfrak{M}$ follows (Morita [5]).

This theorem sharpens theorems of M. Rieffel [6], R. Hart [4], and Faith [3].

We conclude with another application of Theorem 1.

EXAMPLE 3. A Dedekind domain R is integrally closed. For if S is a ring extension of R contained in the quotient field of R, and if S_R is finitely generated, then S_R is a generator by Example 2, so $R \simeq_{\mathrm{nat}} \mathrm{End}_{R'}\, S$. But obviously $R' = \mathrm{End}_R\, S \simeq_{\mathrm{nat}} S$, and $\mathrm{End}_S\, S \simeq_{\mathrm{nat}} S$. Thus $S = R$.

It is remarkable that Theorem 2 should imply both Theorem 3 and Example 3.

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