THE NATURAL METRIC ON $SO(n)/SO(n-2)$ IS THE MOST SYMMETRIC METRIC

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1. **Introduction.** A smooth manifold M admits many different Riemannian structures. The "symmetry" of a fixed Riemannian structure ν on M is usually called the group of isometries of ν, and denoted by $\mathrm{ISO}(\nu)$. It was proved by S. B. Myers and N. E. Steenrod [8] that the group of isometries, $\mathrm{ISO}(\nu)$, is always a Lie group that acts differentiably on M. If M is compact then $\mathrm{ISO}(\nu)$ is also compact. The dimension of $\mathrm{ISO}(\nu)$ provides a rough measure of *the degree of symmetry of the given structure* ν. One is tempted by examples to hope that a Riemannian metric which arises naturally will be the best one in the sense that it is the most symmetric among all possible Riemannian structures. In other words, the isometry group of the natural metric will have bigger dimension than the isometry groups of all other metrics.

The following classical theorem in Riemannian geometry shows that the *natural* metrics on spheres and projective spaces are, indeed, the most symmetric metrics: [1, p. 239].

"If $\dim M=m$, then $\dim \mathrm{ISO}(\nu)\leqq m(m+1)/2$ for any Riemannian metric ν on M; and $\dim \mathrm{ISO}(\nu)=m(m+1)/2$ if and only if $M=S^m$ or P^m and the metric ν is the natural metric."

However, if we look at those classical homogeneous spaces other than spheres and projective spaces, we shall find that the dimension of the isometry group of the natural metric is far less than the bound provided by the above classical theorem. For example, let $M=V_{n,2}=SO(n)/SO(n-2)$, then

$$\dim M=2n-3 \text{ but } \dim \mathrm{ISO}(\nu)=\dim SO(n)\times SO(2)=n(n-1)/2+1$$

for the natural metric ν on $V_{n,2}$. Hence the bound provided by the above theorem, $\frac{1}{2}(2n-3)(2n-2)$, is approximately 4 times bigger than the dimension assumed by the natural one, namely $\frac{1}{2}n(n-1)+1$.

The purpose of this paper is to show that "the natural Riemannian metric on $V_{n,2}$ is indeed the most symmetric metric." We state it more precisely as the following.

THEOREM. *Let $V_{n,2}=SO(n)/SO(n-2)$ be the second Stiefel manifold, with n odd and >20. Then*

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$$\dim \mathrm{ISO}(\nu) \leqq \tfrac{1}{2}n(n-1)+1$$

for every Riemannian metric ν *on* $V_{n,2}$, *and the equality sign holds only when* ν *is the natural metric.*

Since every isometry group, $\mathrm{ISO}(\nu)$, is a compact subgroup of $\mathrm{Diff}(V_{n,2})$, and on the other hand, every compact subgroup of $\mathrm{Diff}(V_{n,2})$ may be realized as a group of isometries with respect to a suitable metric; the above theorem may be restated equivalently as

THEOREM′. *If* G *is a compact subgroup of the group of all diffeomorphisms of* $V_{n,2}$, $\mathrm{Diff}(V_{n,2})$, *n odd and* >20, *then either* $G=\mathrm{SO}(n)$ *or* $\mathrm{SO}(n)\times\mathrm{SO}(2)$ *and is transitive*; *or* $\dim G<n(n-1)/2$ *and is intransitive.*

REMARK. The restriction that n be odd is purely technical. The relative simplicity of the rational homology group of $V_{n,2}$ in the odd case is the main reason. Actually, we strongly believe that a similar theorem should be true for much more general classes of homogeneous spaces. We did not push our method to its limit since we feel that our method may not be suitable to treat the general problem of this type. Our purpose in publishing this note is to show the existence of problems of this kind rather than to demonstrate the method of proof. It is the author's sincere hope that someone will invent new methods for the study of symmetries of geometric structure.

2. **Proof of the theorem.** Our proof heavily relies on the results of [2], [4], [5] and technically we may regard this note just as an application of the results of [2], [4], [7]. In order to apply the results of [4], we need the following proposition:

PROPOSITION 1. *Let* G/H *be a homogeneous space with* $G=G_1\times\cdots\times G_s$ *acting almost effectively on* G/H; *where* G_i *are simple compact Lie groups. Suppose we have*

$$\dim H > r\cdot\dim (G/H) \qquad r>4.$$

Then there is at least one normal factor, say G_1, *that is locally isomorphic to one of the following groups*:

(i) $\mathrm{SO}(l)$; $l>2r+2$ or
(ii) $\mathrm{SU}(l)$; $l>2r+1$ or
(iii) $\mathrm{Sp}(l)$; $l>2r$.

The above proposition may be proved by induction on the number of factors s. A detailed proof may be found in another paper of the author [6].

PROOF OF THE THEOREM. In case G acts transitively on $V_{n,2}$, it was proved in a joint work of J. C. Su and the author that G must be either SO(n) or SO(n)$\times$SO(2) with the usual action [7].

Now, we suppose that G acts intransitively (of course effectively and differentiably). Let G/H be the principal orbit type, then it is clear that G acts effectively on G/H. We shall show that $\dim G < n(n-1)/2$.

Suppose the contrary. Then we have $\dim G \geqq n(n-1)/2$ and $\dim(G/H) \leqq (2n-3)-1=2n-4$. Hence

$$\dim H \geqq \tfrac{1}{2}(n^2-n)-(2n-4) > \tfrac{1}{2}(n-2)(n-4) \geqq \tfrac{1}{4}(n-4) \dim (G/H)$$

where $r=(n-4)/4>4$ for $n>20$. Hence Proposition 1 applies and we have that $G \sim G_1 \times G_2$ with

(i) $G_1 \sim \mathrm{SO}(l)$, $l > \frac{1}{2}n$, or
(ii) $G_1 \sim \mathrm{SU}(l)$, $l > \frac{1}{2}n-1$, or
(iii) $G_1 \sim \mathrm{Sp}(l)$, $l > \frac{1}{2}n-2$.

Since $V_{n,2}$ is a stably parallelizable rational homology sphere of dimension $(2n-3)$, the results of [4] apply for a detailed analysis of the action of G_1 on $V_{n,2}$. We shall divide our argument into three cases according to $G_1 \sim \mathrm{SO}(l)$ or $\mathrm{SU}(l)$ or $\mathrm{Sp}(l)$ respectively.

(i) $G_1 \sim \mathrm{SO}(l)$, $l > \frac{1}{2}n$: In this case, the principal orbit type of the G_1-action on $V_{n,2}$ has three possibilities, namely $\mathrm{SO}(l)/\mathrm{SO}(l-1)$, or $\mathrm{SO}(l)/\mathrm{SO}(l-2)$, or $\mathrm{SO}(l)/\mathrm{SO}(l-3)$. Following an argument of [2], [5] we may show it is impossible to have a compact connected group G acting on $V_{n,2}$, n odd, with S^k, $k \neq 1, 3$, as principal orbit type. Now suppose the principal orbit type of the G_1-action is $\mathrm{SO}(l)/\mathrm{SO}(l-2)$. Then it follows from [4] that the dimension of the fixed point set $\dim F(G_1, V_{n,2}) = (2n-3)-2l \geqq -1$ and the orbit space is naturally a $[(2n-3)-2l+3]$-dimensional manifold X with ∂X being the image of singular orbits. If $F(G_1, V_{n,2}) = \emptyset$, then $l=n-1$ and by considering the action of G_2 on ∂X, we see that $\dim G_2 \leqq 2$. Hence $\dim G = \dim G_1 + \dim G_2 \leqq (n-1)(n-2)/2+2 < n(n-1)/2$, which is a contradiction. If $F(G_1, V_{n,2}) \neq \emptyset$, then G_2 acts on $F(G_1; V_{n,2})$. Let $x \in F(G_1, V_{n,2})$, then $G_x = G_1 \times G_{2_x}$. By looking at the local representation ϕ_x of G_x with the knowledge that $\phi_x | G_1 = 2 \cdot \rho_l \oplus \theta$ $\dim \phi_x = 2n-3$, $l > n/2$, and the representation of G_{2_x} cannot have S^k, $k \neq 1, 3$, as its principal orbit type; we may have a good estimate of $\dim G_{2_x}$. The answer is

$$\dim G_{2_x} \leqq \tfrac{1}{4}(2(n-l)-3)^2.$$

Now,

$$\begin{aligned}\dim G &= \dim G_1 + \dim G_{2_x} + \dim G(x)\\ &\leqq \tfrac{1}{2}l(l-1) + \tfrac{1}{4}(2(n-l)-3)^2 + (2(n-l)-3)\\ &< \tfrac{1}{2}n(n-1) \quad \text{for} \quad n < 2l \leqq 2n-3\end{aligned}$$

which is a contradiction.

The argument for the case that the principal orbit type of the G_1-action is $SO(l)/SO(l-3)$ as well as the cases $G_1 \sim SU(l)$ and $G_1 \sim Sp(l)$ are almost the same as the above case, but are a little bit easier and hence omitted.

From all the above contradictions, we proved the theorem.

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