BOOK REVIEW

Lectures in abstract algebra, Vol. III, *Theory of fields and Galois theory*. By Nathan Jacobson. Van Nostrand, Princeton, N. J., 1964. 323 pp. $9.75.

This book is the third and last of a series of distinguished books by Jacobson which lay out the foundations of abstract algebra. The first two books have already had a wide acceptance and use both as textbooks and source material. It is obvious that this third volume is destined to play the same role. To this reviewer's mind this volume is by far the best and most unusual of the three. In keeping with the style set in all Jacobson's books, be they elementary or very advanced, this book is chock-full of material, often material that is difficult to find elsewhere. As a consequence the book probably will not be easy to read for a casual student, but a serious student who works at the material will get a large pay-off in return for his efforts. There is perhaps a tendency in the book to state results in a very general form which could perplex a person seeing these things for the first time. But these are minor criticisms when confronted with the many, many pluses that the book enjoys.

As its title implies, the book is a study of fields and of Galois theory. The reader expecting the usual Artin treatment of Galois theory has a surprise in store for him—he won't find it here. It is refreshing to see Galois theory handled in a different way. The approach is motivated by the work Jacobson has done on noncommutative Galois theory. Not only does it give the classical commutative field results but it also sets up the machinery to handle the noncommutative situation or to pass on to the kind of work on Galois theory being done now by such people as Chase, Harrison and Rosenberg. It is rooted in the Jacobson-Bourbaki Lemma which describes the commuting ring of a vector space of endomorphisms in the ring of endomorphisms of the additive group of a field. Exploiting this result, the author goes on to develop the Galois theory. In quick order the usual concepts and results come out. Trace and norm make their appearance and there is a short discussion of crossed products and cohomology theory. Having obtained the general results of Galois theory the author sets himself to applying them to the study of the theory of equations. He gets Galois' criterion for solvability by radicals and applies it to obtain Abel's theorem. However, unfortunately no mention is made of the impossibility of trisection or other classical constructibility theorems. (In a sense these belong in more elementary aspects of field theory, namely that degrees of successive extensions multiply, and not in Galois theory. Nonetheless, in my opinion, they should be mentioned in a treatment of field theory.)

It is after this discussion of the usual Galois theory that the book takes on its unique character; for almost everything from this point on is not to be found easily in any one elementary book. There is a real wealth of material here. The tone is no longer that of a textbook. You get the feeling that now we are down to business; this is the real McCoy. And it is. We touch lightly on the many topics treated. He goes into a discussion of Abelian extensions. There is a brief treatment of cyclotomic fields and then a somewhat homological exposition of Kummer fields. Witt vectors then turn up and the structure of the ring of n-Witt vectors given. He closes this material out with a criterion for the existence of cyclic p-extensions of fields of characteristic $p\neq 0$.

Next comes the most impressive chapter of the book, Chapter IV. There is so much wonderful material here that no short description as we give here can do it justice. The existence of the algebraic closure is proved; then some infinite Galois theory. Transcendence basis and degree are studied and Luroth's theorem describing the subfields of the rational function field in one variable is proved. Now comes a thorough discussion of fields of characteristic $p\neq 0$. Everything developed in the book to this point now comes into play and many new techniques (among them Lie algebras and restricted Lie algebras) are presented to give a basic discussion of such things as F. K. Schmidt's theorem on the existence of a basis of an algebraic function field over which the field is separable algebraic, the structure of purely inseparable extensions and composite of fields: this chapter is an amazing masterpiece.

Jacobson next takes up the subject of valuation theory. Here the treatment is a little more standard. Rank one valuations are first defined and the usual things such as completion re a valuation are done. The p-adics are discussed, then the construction of fields complete in a rank 1 valuation with a pre-assigned residue class field. He then defines valuations generally and interrelates them with valuation rings and the notion of place. He treats the Hilbert Nullstellensatz and Ostrowski's theorem about complete fields with real archimedian valuation. The question of the extension of valuations and ramifications are very lightly treated. It is trivial to say that it would have been nice that this or that had been treated; each one has his own idiosyncrasies. I do regret that he did not go into a more thorough exposition of the Hilbert theory of ramification and applications to the structure of local fields and algebraic number fields with, perhaps, some talk of the relation of these topics to division algebras.

The last chapter is a treatment of formally real fields and the results of Artin and Scheier. We say just a few words about this. Here we find a discussion of real closed fields, a solution of the Hilbert problem about a rational function nonnegative on real vectors (for which it is defined) and some results of Tarski and Seidenberg. No proof of the general theorem of Tarski about elementary sentences holding or not holding universally for all real closed fields is given. The book closes with a proof that a field is real closed if its algebraic closure is a finite extension of it.

This is some book, a real contribution to the mathematics literature and a golden opportunity for an aspiring algebraist eager to learn. The whole mathematical community should be grateful for such a distinguished addition to its teaching and learning arsenal.

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