AN UPPER BOUND FOR RAMSEY NUMBERS

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Communicated by V. Klee, July 13, 1966

The finite form of Ramsey's Theorem [2], states that there exists a function $h(n)$ so that if a graph G has at least $h(n)$ points, then either G contains a complete subgraph on n points, or a set of n independent points (a set of points with no edges between any pair). We note that the graphs to be considered have no loops and each pair of points are joined by at most one edge.

Define $f(k, n)$ to be the least integer so that every graph with $f(k, n)$ points contains a complete subgraph on k points or contains a set of n independent points. Erdös and Szekeres [1], proved that

$$f(k, n) \leqq \binom{k+n-2}{k-1}.$$

This upper bound can be improved as we show that

$$f(k, n) \leqq \left(\prod_{i=3}^{k} C_i\right)\frac{n^{k-1}}{(k-1)!} + o(n^{k-1}) \tag{1}$$

where $0<C_i<1$ for $i=3, 4, 5, \cdots$, and in particular

$$f(3, n) \leqq \left(\frac{111 + (33)^{1/2}}{128}\right)\frac{n^2}{2} + o(n^2). \tag{2}$$

DEFINITION 1. A graph G will be called a Ramsey (k, n) graph if it has no complete subgraph on k points and no set of n independent points.

Note that a Ramsey (k, n) graph is not required to have the maximum number, $f(k, n)-1$, of points.

DEFINITION 2. The complement of a point of a graph G is the subgraph of G obtained by deleting from G the given point, all points joined to this point by an edge and all edges incident to these points.

REMARK 1. The complement of a point in a Ramsey (k, n) graph must be a Ramsey $(k, n-1)$ graph. This is obvious since the point is independent of all points in its complement.

REMARK 2. The set of points joined to a given point in a Ramsey (k, n) graph together with their edges must be a Ramsey $(k-1, n)$ graph.

PROOF OF (1). Let G_n be a Ramsey (k, n) graph on N points. Since by Remark 2 each point has valence less than $f(k-1, n)$ we see that the number of edges of G_n is bounded above by

$$\frac{\{f(k-1, n)-1\}N}{2}. \tag{3}$$

We now wish to get a lower bound on the number of edges of G_n and we proceed as follows:

Let P_n be a point of G_n which has minimum valence in G_n. Denote the complement of P_n by G_{n-1}. By Remark 1, G_{n-1} is a Ramsey $(k, n-1)$ graph. We now proceed inductively choosing P_i a point of G_i which has minimum valence in G_i and letting G_{i-1} denote the complement of P_i taken with respect to the graph G_i. Also note that since G_i is a Ramsey (k, i) graph, G_{i-1} will be a Ramsey $(k, i-1)$ graph. This sequence of points $\{P_i\}$ is obviously an independent set of points of G_n hence there are at most $n-1$ points chosen by this process. In our notation we will proceed as though this sequence contained $n-1$ elements.

Let v_i denote the valence of P_i and note that since G_i was a Ramsey (k, i) graph, by Remark 2, $0 \leqq v_i \leqq f(k-1, i)-1$. The number of edges of G_n removed at the ith step of this process can now be estimated. By Remark 2 we see that the points joined to P_i have at most $f(k-2, i)f(k-1, i)/2$ edges between them so that since each point has valence v_i or more we have removed at least

$$v_i^2 - f(k-2, i)f(k-1, i)/2 \text{ edges of } G_n.$$

This gives a lower bound on the number of edges of G_n to be

$$\sum_{i=2}^{n} \left\{ v_i^2 - \frac{f(k-2, i)f(k-1, i)}{2} \right\}. \tag{4}$$

On the other hand we note that the ith step of the process removed exactly v_i+1 points of G_n and (3) can be rewritten as

$$\frac{f(k-1, n)}{2} \sum_{i=2}^{n} (v_i + 1).$$

Combining this with (4) gives us

$$\sum_{i=2}^{n} \left\{ v_i^2 - \frac{f(k-2, i)f(k-1, i)}{2} \right\} \leqq \frac{f(k-1, n)}{2} \sum_{i=2}^{n} (v_i + 1). \tag{5}$$

Since we are interested only in the order of magnitude of these terms we will let

$$v_i = Ci^{k-2} - e_i \tag{6}$$

where $f(k-1, i) \leqq Ci^{k-2} + o(i^{k-2})$. Note that the bound of Erdös and Szekeres guarantees the existence of such a number C. We will show that

$$\sum_{i=2}^{n} e_i = O(n^{k-1})$$

hence this will show the existence of the constant $C_k < 1$.

With this change of notation in (5) and by combining all terms of lower order we assert

$$\sum_{i=2}^{n} (Ci^{k-2} - e_i)^2 \leqq \frac{Cn^{k-2}}{2} \sum_{i=2}^{n} (Ci^{k-2} - e_i) + o(n^{2k-3}). \tag{7}$$

Hence if $\sum e_i = o(n^{k-1})$ we would have

$$\sum_{i=2}^{n} (Ci^{k-2})^2 \leqq \frac{Cn^{k-2}}{2} \sum_{i=2}^{n} (Ci^{k-2}) + o(n^{2k-3})$$

or

$$\frac{n^{2k-3}}{2k-3} \leqq \frac{n^{2k-3}}{2k-2} + o(n^{2k-3}),$$

which is clearly impossible. Hence by (6), $N = \sum Ci^{k-2} - \sum e_i$. Therefore $N \leqq C(C_k)n^{k-1}/k - 1 + o(n^{k-1})$ and (1) follows.

REMARK 3. From (7) and for $k = 3$, where we know $C = 1$, we have

$$\sum_{i=2}^{n} (i - e_i)^2 \leqq \frac{n}{2} \sum_{i=2}^{n} (i - e_i) + o(n^3)$$

which easily shows that

$$C_3 \leqq \frac{111 + (33)^{1/2}}{128}.$$

REMARK 4. Estimation of the constant C_k, $k = 3, 4, 5, \cdots$ shows that

$$2\left(1 - \frac{1}{k}\right)^{k-1} \leqq C_k \leqq \left(1 - \frac{1}{k^2}\right)^{k-2} \frac{(k+2)(k-1)}{k^2}.$$

REFERENCES

1. P. Erdös and G. Szekeres, *On a combinatorial problem in geometry*, Composition Math. 2 (1935), 463–470.

2. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. (2) **30** (1930), 264–286.

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COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

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Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [**1**] that at least n nth powers are required to sum to an nth power, $n > 2$.

REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.