ZEROS AND FACTORIZATIONS OF HOLOMORPHIC FUNCTIONS

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For $N=1, 2, 3, \cdots$ we let U^N denote the Cartesian product of N copies of the open unit disc U. I.e., U^N consists of all $z=(z_1, \cdots, z_N)$ in C^N (the space of N complex variables) with $|z_j|<1$ for $j=1, \cdots, N$. We write U in place of U^1. If $1 \leqq p < \infty$, $H^p(U^N)$ is the space of all holomorphic functions f in U^N for which

$$\sup (1/2\pi)^N \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(r_1e^{i\theta_1}, \cdots, r_Ne^{i\theta_N})|^p d\theta_1 \cdots d\theta_N < \infty,$$

the supremum being taken over all choices of $r_1, \cdots, r_N$ such that $0 \leqq r_j < 1$. The pth root of this supremum is defined to be $\|f\|_p$; this gives a Banach space norm. (The boundary behavior of these functions is discussed in Chapter XVII of [**3**].)

The class of all bounded holomorphic functions in U^N is denoted by $H^\infty(U^N)$.

The *zero-set* of a function f defined in U^N is the set of all $z \in U^N$ at which $f(z)=0$.

It is well known that the zero-set of every $f \in H^p(U)$, for any p, is also the zero-set of some $g \in H^\infty(U)$. These zero-sets, in one variable, are completely characterized by the Blaschke condition $\sum(1-|\alpha_i|) < \infty$. For $N>1$ a different phenomenon occurs:

THEOREM A. *There exists a function f, not identically* 0, *such that*
(a) $f \in H^p(U^2)$ *for all* $p<\infty$, *but*
(b) *if $g \in H^\infty(U^2)$ and if the zero-set of g contains the zero-set of f, then g is identically* 0.

Let us call a subspace S of $H^p(U^N)$ *invariant* if multiplication by the coordinate functions $z_1, \cdots, z_N$ maps S into S. The closed invariant subspaces of $H^p(U)$ are known precisely: they are generated by inner functions [1, pp. 8, 25]. But if we consider the smallest closed invariant subspace of $H^p(U^2)$ which contains the function f of Theorem A we obtain the following:

COROLLARY. *If $1 \leqq p < \infty$, there is a nontrivial closed invariant subspace of $H^p(U^2)$ which contains no bounded function* (*except* 0).

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To every $f \in H^1(U)$ there correspond two functions g, $h \in H^2(U)$ such that $f \in gh$. The usual proof of this factorization theorem [**3**, Vol. I, p. 275] even shows that g and h can be so constructed that their boundary values satisfy $|g|^2 = |h|^2 = |f|$ a.e. The work of Helson and Lowdenslager has extended this stronger result to H^1-functions on compact connected abelian groups G, where analyticity is defined relative to some total order of the dual group of G [**2**, p. 208]. It seems likely that the factorization fails in $H^1(U^N)$ if $N > 1$. The following theorem shows at least that the above-mentioned stronger result fails very badly if $N > 2$.

THEOREM B. *Suppose* $\epsilon > 0$, $M < \infty$. *There exists an irreducible homogeneous polynomial* f *in* 3 *variables, with* $\|f\|_1 < \epsilon$, $\|f\|_2 > M$. *For any such* f *we have* $\|g\|_2 \|h\|_2 > M$ *whenever* $f = gh$ *and* g, $h \in H^2(U^3)$.

An immediate consequence of Theorem B is the observation that the bilinear continuous map

$$\mu \colon H^2(U^3) \times H^2(U^3) \to H^1(U^3),$$

defined by $\mu(g, h) = gh$, is not open at the origin. This by itself may imply that the range of μ cannot be all of $H^1(U^3)$.

In any case, Theorem B suffices to establish a "nonfactorization theorem" for Dirichlet series. Let us say that a function F of the form

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is a *Dirichlet series of class* H^p $(1 \leq p < \infty)$ if (a) the series converges absolutely in the open right half-plane, and (b)

$$\sup_{\sigma > 0} \left\{ \lim_{T \to \infty} 1/2T \int_{-T}^{T} | F(\sigma + it) |^p dt \right\} < \infty .$$

(The existence of the limit is assured by almost periodicity.)

THEOREM C. *There exists a Dirichlet series of class* H^1 *which is not a product of two Dirichlet series of class* H^2.

We conclude with brief outlines of the proofs.

PROOF OF THEOREM A. Fix a number $R > 1$. Let $\{\alpha_k\}$ be a sequence of complex numbers, $|\alpha_k| = 1$, such that every point of some infinite set E occurs infinitely many times in $\{\alpha_k\}$. Let $\{n_k\}$ be a rapidly increasing sequence of positive integers. Define

$$f(z, w) = \prod_{k=1}^{\infty} \left\{ 1 - R\left(\frac{z + \bar{\alpha}_k w}{2} \right)^{n_k} \right\} .$$

Note that $|z+\bar{\alpha}_k w| \leqq 2$ on the closure of U^2 and that equality occurs only on a circle in the distinguished boundary T^2, i.e., on a set of measure 0. Hence $\{n_k\}$ can be chosen inductively so that the integrals $\int_{T^2}|f_m|^p$ are bounded, as $m \to \infty$, for each $p < \infty$ (the bound will depend on p), where f_m denotes the product of the first m factors. The product will also converge uniformly on compact subsets of U^2. Thus $f \in H^p(U^2)$ and f does not vanish identically.

Suppose $g \in H^\infty(U^2)$ and $g=0$ whenever $h=0$. For $\beta \in E$ and $\lambda \in U$ put $g_\beta(\lambda) = g(\lambda, \beta\lambda)$. If $\alpha_k = \beta$, then g_β vanishes at every n_kth root of $1/R$. This happens for infinitely many k, and since $g_\beta \in H^\infty(U)$ one deduces from Jensen's formula that $g_\beta(\lambda)=0$ for all $\lambda \in U$. In other words, the zero-set of g contains every disc

$$D_\beta = \{(\lambda, \beta\lambda) : \lambda \in U\} \quad (\beta \in E).$$

All these discs intersect at (0, 0). This forces g to be identically 0.

PROOF OF THEOREM B. Let P_n be the space of all homogeneous polynomials of degree n, in 3 variables. If n is large enough, there exists $f \in P_n$ with $\|f\|_1 < \epsilon$, $\|f\|_2 > M$. A dimensionality argument shows that the irreducible members of P_n form a dense (in fact, open) subset of P_n. Hence we can adjust f so that it is irreducible. If now $f = gh$, $g = \sum g_k$, $h = \sum h_k$, where g_k and h_k are homogeneous polynomials of degree k, then f is the product of the lowest nonvanishing components of g and h, say $f = g_j h_{n-j}$. But f is irreducible. Hence $j=0$ or $j=n$. Finally, $\|g\|_2 \geqq \|g_j\|_2$, since the various g_k's are orthogonal to each other; likewise, $\|h\|_2 \geqq \|h_{n-j}\|_2$.

PROOF OF THEOREM C. There are homogeneous irreducible polynomials $f_k(z_1, z_2, z_3)$ with $\|f_k\|_1 < 2^{-k}$, $\|f_k\|_2 > k$. Let N_k be the degree of f_k, let C_k be the sum of the absolute values of the coefficients of f_k, let $\{p_j\}$ be an increasing sequence of distinct primes such that

$$p_{3k} > (k^2 C_k)^{k/N_k},$$

and define

$$F(s) = \sum_{k=1}^{\infty} f_k(p_{3k}^{-s}, p_{3k+1}^{-s}, p_{3k+2}^{-s}).$$

Our choice of $\{p_j\}$ assures the absolute convergence of the Dirichlet series of $F(s)$, if Re $s > 0$. With the aid of Theorem B it follows easily that this F has the properties stated in Theorem C. In fact, one can even show that F is not the product of any finite number of Dirichlet series of class H^2.

References

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2. Walter Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.

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