AUTOMORPHISMS OF OPERATOR ALGEBRAS

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1. **Introduction.** We describe certain results concerning the structure of the group $\alpha(\mathfrak{A})$ of automorphisms of a C^*-algebra $\mathfrak{A}$. They will appear, together with their proofs, in Communications in Mathematical Physics.

All mappings of $\mathfrak{A}$ we consider preserve the * structure. With ϕ a faithful representation of $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$, we say that an automorphism β of $\phi(\mathfrak{A})$ is weakly-inner when there is a unitary operator U in the strong-operator closure $\phi(\mathfrak{A})^-$ of $\phi(\mathfrak{A})$ such that $\beta(A) = UAU^*$ for all A in $\phi(\mathfrak{A})$. We denote by $\iota_\phi(\mathfrak{A})$ the subgroup of $\alpha(\mathfrak{A})$ consisting of those α for which $\phi\alpha\phi^{-1}$ is weakly-inner, by $\epsilon_\phi(\mathfrak{A})$ those α such that $\phi\alpha\phi^{-1}$ extends to an automorphism of $\phi(\mathfrak{A})^-$, by $\sigma_\phi(\mathfrak{A})$ those α such that there is some unitary operator U on $\mathcal{H}$ for which $\phi\alpha\phi^{-1}(A) = UAU^{-1}$ when A lies in $\phi(\mathfrak{A})$ and by $\pi(\mathfrak{A})$ the intersection of all $\iota_\phi(\mathfrak{A})$. We refer to the elements of $\sigma_\phi(\mathfrak{A})$, $\epsilon_\phi(\mathfrak{A})$ and $\pi(\mathfrak{A})$ as the *spatial*, *extendable*, *permanently weakly* (for brevity, $\pi-$) *inner* automorphisms, respectively, (of $\phi(\mathfrak{A})$ in the first two instances and $\mathfrak{A}$ in the last). We denote by $\iota_0(\mathfrak{A})$ the group of inner automorphisms of $\mathfrak{A}$.

The group $\alpha(\mathfrak{A})$ consists of operators on the Banach space $\mathfrak{A}$ each of which is isometric so that $\alpha(\mathfrak{A})$ acquires a norm (or metric) topology as a subset of the bounded operators on $\mathfrak{A}$. We consider $\alpha(\mathfrak{A})$ and the various subgroups defined as provided with this topology. We denote by $\gamma(\mathfrak{A})$ the connected component of the identity element ι of $\alpha(\mathfrak{A})$.

2. **The automorphism group.** It has been proved recently [3], [4], [5] that each derivation δ of a C^*-algebra $\mathfrak{A}$ acting on the Hilbert space $\mathcal{H}$ is weakly inner—that is, there is a B in $\mathfrak{A}^-$ such that $\delta(A) = BA - AB$ for all A in $\mathfrak{A}$. Combining this result with those of Nagumo-Yosida and some series computations, we have:

LEMMA 1. *Each norm-continuous one-parameter subgroup of* $\alpha(\mathfrak{A})$ *lies in* $\pi(\mathfrak{A})$.

Spectral theory and von Neumann algebra considerations yield:

LEMMA 2. *If α is an inner automorphism of a von Neumann algebra*

[1] Completed with the partial support of NSF and ONR.

$\mathcal{R}$ such that $\|\alpha-\iota\|<2$; then there is a unitary operator U in $\mathcal{R}$ with spectrum $\sigma(U)$ in the half-plane $\{z\colon \operatorname{Re} z \geqq \frac{1}{2}(4-\|\alpha-\iota\|^2)^{1/2}\}$ such that $\alpha(A)=UAU^*$ for all A in $\mathcal{R}$.

Geometrically, $\sigma(U)$ is contained in the arc of the unit circle containing 1 with endpoints midway between 1 and the points of the circle at distance $\|\alpha-\iota\|$ from 1. Employing the automorphism β of the factor of type II_1 associated with the free group on three generators arising from a cyclic permutation of these generators, we see that the conclusion of the lemma fails in this case ($\|\beta-\iota\|=2$).

Disjoint representation together with von Neumann algebra methods permit us to prove:

LEMMA 3. *If α is an automorphism of the C^*-algebra $\mathfrak{A}$ acting on $\mathcal{H}$ such that $\|\alpha-\iota\|<2$, then α extends to an automorphism $\bar{\alpha}$ of $\mathfrak{A}^-$ such that $\|\bar{\alpha}-\iota\|=\|\alpha-\iota\|<2$.*

Techniques of the theory of operator-valued analytic functions combined with Banach algebra methods (much like those of [2], or directly, using [2; Corollary 3]) give:

LEMMA 4. *If $\mathfrak{A}$ is a C^*-algebra and U a unitary operator acting on $\mathcal{H}$ such that $\alpha(A)=UAU^*$ lies in $\mathfrak{A}$ for all A in $\mathfrak{A}$ and $\operatorname{Re}\lambda>0$ for each λ in $\sigma(U)$ then α lies on a norm-continuous one-parameter subgroup of $\alpha(\mathfrak{A})$ and is π-inner.*

The outer automorphism of the II_1 factor associated with the free group on two generators and the unitary operator (multiplied by $(-1)^{1/2}$) arising from interchanging those generators illustrate the fact that the conclusion of the above lemma need not hold if the hypothesis is weakened to allow $\sigma(U)$ to lie in the *closed* right half-plane. Note that Lemmas 2 and 4 allow us to conclude that Lemma 2 holds with "inner" deleted and $\mathcal{R}$ a C^*-algebra (replacing the second occurrence of $\mathcal{R}$ by $\mathcal{R}^-$).

The preceding results and some representation theory for C^*-algebras allow us to conclude:

THEOREM 5. *If α is an automorphism of a C^*-algebra $\mathfrak{A}$ and $\|\alpha-\iota\|<2$ then α lies on a norm-continuous one-parameter subgroup of $\alpha(\mathfrak{A})$. Such subgroups generate $\gamma(\mathfrak{A})$ (as a group); and $\gamma(\mathfrak{A})$ is an open subgroup of $\alpha(\mathfrak{A})$, each element of $\gamma(\mathfrak{A})$ being π-inner. Since*

$$\gamma(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \iota_\phi(\mathfrak{A}) \subseteq \sigma_\phi(\mathfrak{A}) \subseteq \epsilon_\phi(\mathfrak{A}) \subseteq \alpha(\mathfrak{A}),$$

all these subgroups are open as well as closed in $\alpha(\mathfrak{A})$. Moreover, $\gamma(\mathfrak{A})$, $\pi(\mathfrak{A})$ and $\iota_0(\mathfrak{A})$ are normal subgroups of $\alpha(\mathfrak{A})$; while $\iota_\phi(\mathfrak{A})$, $\sigma_\phi(\mathfrak{A})$ and

$\epsilon_\phi(\mathfrak{A})$ *need not be. In addition* $\iota_\phi(\mathfrak{A})$ *is a normal subgroup of* $\epsilon_\phi(\mathfrak{A})$, *though* $\sigma_\phi(\mathfrak{A})$ *need not be.*

As an immediate consequence, we have:

COROLLARY 6. *Each (norm) continuous representation of a connected topological group in* $\alpha(\mathfrak{A})$ *has range consisting of* π*-inner automorphisms of* $\mathfrak{A}$.

COROLLARY 7. *If* $\mathfrak{A}$ *has a faithful representation* ϕ *as a von Neumann algebra then* $\iota_0(\mathfrak{A})=\gamma(\mathfrak{A})=\pi(\mathfrak{A})=\iota_\phi(\mathfrak{A})$; *and each element of* $\gamma(\mathfrak{A})$ *lies on some norm-continuous one-parameter subgroup of* $\alpha(\mathfrak{A})$.

The situation described by the results of this section is sharply in contrast with that which prevails when $\alpha(\mathfrak{A})$ is provided with topologies weaker than its norm topology. In [1] Blattner shows that each locally compact group satisfying a countability axiom has a strongly-continuous unitary representation by operators inducing outer automorphisms of a factor of type II_1 (except for the unit element).

3. **Examples.** The examples which follow illustrate, for specific C^*-algebras, various automorphism phenomena not described by the results of §2.

(a) If $\mathfrak{A}$ has a norm-dense subalgebra which is the ascending union of * subalgebras (with the unit of $\mathfrak{A}$) $\mathfrak{M}_n$ isomorphic to the algebra of operators on 2^n-dimensional Hilbert space, the automorphism α of $\mathfrak{A}$ which is (successive) transposition about both diagonals of each $\mathfrak{M}_n$ is weakly-inner in some (irreducible) representation of $\mathfrak{A}$ on $\mathfrak{L}_2(0, 1)$ provided with Lebesgue measure with [0, 1] partitioned by dyadic rationals while it is not extendable (and *a fortiori* not weakly-inner) in some (irreducible) representation of $\mathfrak{A}$ on $\mathfrak{L}_2([0, 1), \mu)$, where $\mu(S)$ is the number of dyadic rationals in S and [0, 1) is partitioned by left-closed, right-open dyadic rational subintervals. In this case, some $\iota_\phi(\mathfrak{A})$ is distinct from $\pi(\mathfrak{A})$. This can be used to show that $\iota_\psi(\mathfrak{A}_0)$ and $\epsilon_\psi(\mathfrak{A}_0)$ are not normal in $\alpha(\mathfrak{A}_0)$ for some faithful representation ψ of $\mathfrak{A}_0(=\mathfrak{A}\oplus\mathfrak{A})$.

(b) The sum of the scalars with the compact operators gives an example of a C^*-algebra with unit for which all automorphisms are weakly-inner (in the infinite-dimensional irreducible representation) but many are not inner. In this case, $\gamma(\mathfrak{A})=\alpha(\mathfrak{A})$.

(c) With $\mathfrak{M}$ a factor of type II_1 having coupling 1 acting on separable Hilbert space and $\mathfrak{C}$ the compact operators, the linear span of $\mathfrak{M}$ and $\mathfrak{C}$ is a C^*-algebra $\mathfrak{A}$. Denoting the given representation of $\mathfrak{A}$ by ϕ, we have $\iota_\phi(\mathfrak{A})=\alpha(\mathfrak{A})$. Each nonscalar unitary operator U' in

$\mathfrak{M}'$ such that $\|U'-I\|<1$ induces an automorphism α of $\mathfrak{A}$ for which $\|\alpha-\iota\|<2$; so that α is π-inner, yet α is not inner.

(d) Denoting the algebra of $n\times n$ matrices by $\mathfrak{M}_n$ and by $C(X)$ the algebra, under pointwise operations, of complex-valued continuous functions on the compact-Hausdorff space X, provides us with a class of C^*-algebras $C(X)\otimes\mathfrak{M}_n(=\mathfrak{A})$ for which $\gamma(\mathfrak{A})\subseteq\iota_0(\mathfrak{A})\subseteq\pi(\mathfrak{A})\subseteq\alpha(\mathfrak{A})$. The group of automorphisms of $\mathfrak{A}$ which leave its center $C(X)\otimes\{\lambda I\}$ elementwise fixed coincides with $\pi(\mathfrak{A})$. Moreover $\pi(\mathfrak{A})$ is isomorphic to the group of continuous mappings of X into $\alpha(\mathfrak{M}_n)$ (which is $U(n)/T_1$, where $U(n)$ is the n-dimensional unitary group and T_1, its center, is the circle group). The group of mappings which "lift" from $U(n)/T_1$ to the bundle space $U(n)$ coincides with $\iota_0(\mathfrak{A})$; and $\gamma(\mathfrak{A})$ is the subgroup consisting of those mappings into $U(n)/T_1$ homotopic to the constant mapping (onto the unit T_1). Thus $\pi(\mathfrak{A})/\gamma(\mathfrak{A})$ is the group of homotopy classes of mappings of X into $U(n)/T_1$ and $\iota_0(\mathfrak{A})/\gamma(\mathfrak{A})$ those classes which can be lifted to $U(n)$.

Since $U(n)/T_1$ is isomorphic to $SU(n)/\boldsymbol{Z}_n$, where $\boldsymbol{Z}_n$, the center of $SU(n)$, is the group of nth roots of unity, $\pi_1(U(n)/T_1)\approx\boldsymbol{Z}_n$; and the bundle $\{U(n), p, U(n)/T_1, T_1, T_1\}$ (p the natural projection of $U(n)$ onto $U(n)/T_1$) does not have a cross section (since $\pi_1(U(n))\approx\boldsymbol{Z}$). Taking X as $U(n)/T_1$, the identity mapping cannot be lifted so that $\iota_0(\mathfrak{A})\neq\pi(\mathfrak{A})$. The mapping $U\boldsymbol{Z}_n\rightarrow U^nT_1$ of $SU(n)/\boldsymbol{Z}_n$ into $U(n)/T_1$ can be shown to be essential by using $\pi_3(U(n))\approx\boldsymbol{Z}$ and the exactness of the homtopy sequence of our bundle. Thus $\gamma(\mathfrak{A})\neq\iota_0(\mathfrak{A})$.

With X taken as T_1, each mapping of X into $U(n)/T_1$ can be lifted to $U(n)$ since the fibre is arcwise connected; so that $\iota_0(\mathfrak{A})=\pi(\mathfrak{A})$. Using a generator of $\pi_1(U(n)/T_1)$, we have an essential mapping of X into $U(n)/T_1$, so that $\gamma(\mathfrak{A})\neq\iota_0(\mathfrak{A})$.

Taking X to be the 2-skeleton of a simplicial decomposition of $U(2)/T_1$ (homeomorphic to projective 3-space), we have $H^1(X, \boldsymbol{Z})$ is 0; so that each mapping of X into T_1 is inessential (as is each mapping of X into S^3). Thus each mapping of X into $U(2)$ (homeomorphic to $T_1\times S^3$) is inessential, and $\iota_0(\mathfrak{A})=\gamma(\mathfrak{A})$. However, the identity mapping of X onto itself does not lift since $\pi_2(T_1)=0$, and there would be no obstruction to extending this cross section over X to one over $U(2)/T_1$ (while we have seen that these bundles do not have cross sections).

Denoting by $\mathfrak{A}_{m,n}$ the C^*-algebra $C(S^m)\otimes\mathfrak{M}_n$, universal bundle techniques allow us to show that $\iota_0(\mathfrak{A}_{m,n})=\pi(\mathfrak{A}_{m,n})$ for all m, $n=1, 2, \cdots$; and Bott's periodicity theorem tells us that $\gamma(\mathfrak{A}_{m,n})=\pi(\mathfrak{A}_{m,n})$ for even $m<2n$ while $\pi(\mathfrak{A}_{m,n})/\gamma(\mathfrak{A}_{m,n})\approx\boldsymbol{Z}$ for odd $m\neq 1$ less than $2n$.

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