THE HITTING CHARACTERISTICS OF A STRONG MARKOV PROCESS, WITH APPLICATIONS TO CONTINUOUS MARTINGALES IN R^n

BY G. E. DENZEL

Communicated by M. Loeve, July 11, 1966

1. **Introduction.** M. Arbib showed in [1] that, essentially, on the real line a continuous path process with the same "hitting characteristics" as a diffusion was itself a diffusion (strong Markov process). His methods did not lend themselves to more general processes. The purpose of this note is to give a general characterization along this line, for right continuous, nonterminating, quasi-left continuous strong Markov processes with left limits, taking their values in a locally compact, noncompact second countable space E. We also give some interesting consequences concerning continuous martingales in R^n. Full proofs of these and related results will appear elsewhere.

2. **Hitting characteristics.** Let $\hat{X}$ be a process as above, described by measures $\hat{P}^x$ $(x \in E)$ on the space of paths (assume that the function $x \to \hat{P}^x(A)$ is Borel measurable for all Borel $A \subset E$, and that $\hat{P}^x$ $(X_0 = x) = 1$). Let F_t be the σ-field generated by the path functions X_s $(s \leq t)$, and let F_R^+ be the σ-field of the stopping time R ($A \in F_R^+$ if $A \cap \{R < t\} \in F_t$ for all t). Another process X, described by a measure P on the same path space, will be said to have the same *hitting characteristics* as $\hat{X}$ if

$$\text{(H1)} \qquad E[T_{\bar{G}^c} \circ \theta_R \mid F_R^+] = \hat{E}^{X_R}[T_{\bar{G}^c}] \quad \text{P-a.s.,}$$

$$\text{(H2)} \qquad E[I_B \circ X_{T_{\bar{G}^c}} \circ \theta_R \mid F_R^+] = E^{X_R}[I_B \circ X_{T_{\bar{G}^c}}] \quad \text{P-a.s.}$$

for every stopping time R, Borel set $B \subset E$, and open set $G \subset E$ with compact closure ($T_{\bar{G}_n^c}$ is the first hitting time of the complement of the closure of G, and θ_R is the shift by R).

We write μ for the distribution of X_0 under P.

THEOREM 1. *If X and $\hat{X}$ have the same hitting characteristics as described above, and if there is a sequence of sets $G_n \nearrow E$, G_n open with compact closure, such that $x \to \hat{E}^X[T_{\bar{G}_n^c}]$ is a bounded function on E, then $P = \hat{P}^\mu$—i.e., X is a strong Markovprocess.*

The existence of the sets G_n follows whenever $\hat{X}$ is, say, a Feller process.[1] The proof of Theorem 1 is by standard techniques from the theory of Markov process.

3. **Consequences.** Arbib used his theorem to generalize Levy's martingale characterization of Brownian motion [2] to other diffusions on the real line. We similarly use ours to obtain the following theorem.

THEOREM 2. *If* (X_t) *is a process in* R^n *such that* $(h \circ X_t)$ *is a continuous local martingale for every spherical harmonic polynomial* h, *and if* $|X_t|^2/n - t$ *is a continuous local martingale, then* (X_t) *is a Brownian motion.*

Dambis [3] proved that on the line any continuous martingale (X_t) with $X_0 = 0$ was equivalent to a continuous random time change of Brownian motion. Using a slight generalization of his methods, and our Theorem 2, we obtain the following result.

THEOREM 3. *If* (X_t) *is a process in* R^n *with* $h \circ X_t$ *a continuous local martingale for every spherical harmonic polynomial* h, *and if* $X_0 = 0$, *then* (X_t) *is equivalent to a continuous random time change of Brownian motion.*

A slightly weaker version of this last result was obtained by Dubins and Schwarz [4] and by Kunita and Watanabe [5], both using much different techniques. The latter paper also contains a different proof of Theorem 2.

REFERENCES

1. M. Arbib, *Hitting and martingale characterizations of one-dimensional diffusions*, Z. Wahrsch. **3** (1965), 232–247.

2. P. Lèvy, *Processus stochastiques et mouvement Brownien*, Gauthier-Villars, Paris, 1948, p. 78.

3. K. E. Dambis, *On the decomposition of continuous submartingales*, Teor. Verojatnost. i Primenen. **10** (1965), 438–448. (Russian)

4. L. E. Dubins and Gideon Schwarz, *On continuous martingales*, Proc. Nat. Acad. Sci. **53** (1965), 913–916.

5. Hiroshi Kunita and Shinzo Watanabe, *On square integrable martingales*, Privately communicated preprint, 1966.

DARTMOUTH COLLEGE

[1] (On a space E such that $\hat{P}^x[T_{K^c} < \infty] > 0 \ \forall x \in E$, compact $K \subset E$. This might be called the "natural" state space.)