LINEAR FUNCTIONALS ON THE SPACE OF QUASI-CONTINUOUS FUNCTIONS

BY JAMES A. RENEKE

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Suppose that S is a number interval and J is a nondecreasing sequence of closed and compact number intervals with limit S. Let G denote the space of all quasi-continuous functions from S into the plane. If A is a set then 1_A will denote the characteristic function of A. Let Ω denote the collection of subsets of S to which A belongs only in case $1_A G$ is contained in G. J has final set in Ω. For each integer n let $|\cdot|_n$ denote the norm for $1_{J(n)}G$ defined by $|f|_n = \text{l.u.b. } |f(x)|$ for all x in $J(n)$. Let $|\cdot|$ denote the function from G to the nonnegative numbers defined by

$$|f| = \sum_{p=1}^{\infty} 2^{-p} |1_{J(p)}f|_p/(1 + |1_{J(p)}f|_p).$$

G is complete in the topology generated by the metric $\rho(f, g) = |f-g|$ and $1_{J(n)}G$ is a closed linear subspace of G for each positive integer n. A linear functional F on G is continuous only in case the restriction of F to $1_{J(n)}G$ is continuous with respect to $|\cdot|_n$ for each positive integer n.

THEOREM. *For each continuous linear functional F on G there is an ordered triple $\{U, V, W\}$ of order additive functions from $S \times S$ to the plane such that if A is in Ω, A is contained in $[a, b]$, and a is not in A, then*

$$F(f) = (L)\int_a^b fU + (I)\int_a^b f(-U + V - W) + (R)\int_a^b fW$$

for each f in $1_A G$. Furthermore, if u is an increasing function from $[a, b]$ such that

(1) $U(s-, s) = V(s-, s) = W(s-, s) = 0$, *when s is in $(a, b]$ and* $u(s) = u(s-)$,

and

(2) $U(s, s+) = V(s, s+) = W(s, s+) = 0$, *when s is in $[a, b)$ and* $u(s) = u(s+)$,

and v denotes the function from $[a, b]$ defined by

$$v(s) = -(R)\int_s^b duU[\,,b] + (Y)\int_s^b du(U[\,,b] - V[\,,b] + W[\,,b]) - (L)\int_s^b duW[\,,b];$$

then $F(f) = \int_a^b df dv/du$ *for each* f *in* $1_A G$.

PROOF. The proof depends on James R. Webb's idea for using F to define a class of order additive functions from $S\times S$ to the conjugate space of G and J. S. Mac Nerney's representation of an integral as the sum of a left, a right, and an interior integral. We will assume Mac Nerney's definitions and notation as given in [3]. Let $\mathcal{O}\mathcal{B}$ denote the space of functions from S to the plane which have bounded variation on each compact subinterval of S. $\mathcal{O}\mathcal{B}$ is contained in G. Let $\mathcal{O}\mathcal{A}$ denote the class of order additive functions from $S\times S$ to the plane to which V belongs only in case there is an order additive function α from $S\times S$ to the numbers such that $|V(x, y)| \leqq \alpha(x, y)$ for each $\{x, y\}$ in $S\times S$.

For each B in Ω let F_B denote the linear functional on G defined by $F_B(f) = F(1_B f)$. Let K denote the function from $\mathcal{O}\mathcal{B}$ to the order additive functions from $S\times S$ defined by

$$\begin{aligned} Kf(s, t) &= F_{(s,t]}(f) && \text{if } s < t \\ &= 0 && \text{if } s = t \\ &= -F_{(t,s]}(f) && \text{if } s > t. \end{aligned}$$

If each of n and m is a positive integer and $(s, t]$ is a subinterval of S which is contained in $J(n)$ then $|1_{(s,t]}f|_n = |1_{(s,t]}f|_{n+m}$ for each f in G and so $\|F_{(s,t]}\|_n = \|F_{(s,t]}\|_{n+m}$, where $\|\cdot\|_n$ denotes the norm for the conjugate space of $1_{J(n)}G$ corresponding to $|\cdot|_n$. If $s<r<t$ then

$$\|F_{(s,r]}\|_n + \|F_{(r,t]}\|_n = \|F_{(s,t]}\|_n$$

[5, Lemma 3.9]. Let λ denote the function $S\times S$ to the nonnegative numbers defined as follows: if s is in S then $\lambda(s, s) = 0$, and if s and t are in S and $s<t$ then $\lambda(s, t) = \lambda(t, s) = \text{l.u.b.}\ \|F_{(s,t]}\|_n$ for $n = 1, 2, \cdots$.

λ is order additive and if f is in G, n is a positive integer, $[s, t]$ is a subinterval of S contained in $J(n)$, and b is a number such that $|f(x)| \leqq b$ for each x in $[s, t]$, then $Kf(s, t) \leqq \lambda(s, t)b$. Thus K satisfies Mac Nerney's Axioms I and II [3, p. 321] and his representation theorem establishes the existence of an ordered triple $\{U, V, W\}$ of functions in $\mathcal{O}\mathcal{A}$ such that

$$Kf(s, t) = (L)\int_s^t fU + (I)\int_s^t f(-U + V - W) + (R)\int_s^t fW$$

for each f in $\mathfrak{O}\mathfrak{B}$ and $\{s, t\}$ in $S\times S$.

If A is in Ω, A is contained in $[a, b]$, and a is not in A, then $F(f) = Kf(a, b)$ for each f in the common part of $\mathfrak{O}\mathfrak{B}$ and $1_A G$. Since the common part of $\mathfrak{O}\mathfrak{B}$ and $1_A G$ is dense in $1_A G$ [1],

$$F(f) = (L)\int_a^b fU + (I)\int_a^b f(-U + V - W) + (R)\int_a^b fW$$

for each f in $1_A G$. If f is in $1_A G$, c is a number, $g = 1_{(c,\infty)}f$, and $h = 1_{[c,\infty)}f$, then integration by parts [4] yields

$$F(g) = \int_a^b dg dv/du \quad \text{and} \quad F(h) = \int_a^b dh dv/du.$$

Hence $F(f) = \int_a^b df dv/du$ for each f in $1_A G$ [1, Lemma 4.1b].

REMARK. H. S. Kaltenborn [1] obtained representations of continuous linear functionals on $1_{[a,b]}G$ in terms of mean, interior, and Young integrals, but always with a remainder term. Webb, using different methods, obtained representations of continuous linear functionals on $1_{(a,b]}G$ as Hellinger integrals.

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NEWBERRY COLLEGE, CLEMSON UNIVERSITY