ROYDEN'S MAP BETWEEN RIEMANN SURFACES[1]

BY MITSURU NAKAI

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Let R and R_j $(j=1, 2)$ be Riemann surfaces, either open or closed. We denote by $M(R)$ *Royden's algebra* associated with R, and by R^* *Royden's compactification* of R (see [5], [6], and [7]). We have seen in [5] that every algebraic isomorphism of $M(R_1)$ onto $M(R_2)$ induces (and is induced by) a quasiconformal mapping of R_1 onto R_2. In other words, the algebraic structure of $M(R)$ characterizes the quasiconformal structure of R. In this connection there naturally arises the following question: What can we say about the topological structure of R^*? This question leads us to a new notion, Royden's map, which seems to be of considerable function-theoretic interest.

Here we report, without proofs, some of the properties of Royden's maps. Details will be published elsewhere.

1. **Moduli of A-sets.** An open subset G of R is called *normal* if for any point z in ∂G there exists a parametric disk U, with center z, such that $\partial G \cap U$ is a simple arc connecting two boundary points of U.

An *A-set* A is a pair (G_1, G_2) of two nonempty normal open sets G_1 and G_2 in R with $G_1 \supset \overline{G}_2$. An annulus in a parametric disk is an example of an A-set.

We associate with an A-set $A=(G_1, G_2)$ a family $\{\phi\}$ of functions ϕ which are continuous on $\overline{G}_1 - G_2$, of class C^1 in $G_1 - \overline{G}_2$, and have boundary values $\phi | \partial G_j = j$ $(j=1, 2)$. The *modulus* of A, denoted mod A, is the number in $[0, \infty)$ given by

$$(1) \qquad \text{mod } A = 2\pi / \inf_{\phi \in \{\phi\}} D(\phi),$$

where $D(\phi)$ is the Dirichlet integral of ϕ taken over $G_1 - \overline{G}_2$. If A is an annulus in a parametric disk, then this definition coincides with the usual one.

2. **Royden's map.** A topological mapping T of R_1 onto R_2 carries an A-set $A=(G_1, G_2)$ on R_1 to the A-set $TA=(TG_1, TG_2)$ on R_2. We call T a *Royden's map* if there exists a constant $K(A) \geqq 1$ such that

$$(2) \qquad K(A)^{-1} \text{ mod } A \leqq \text{mod } TA \leqq K(A) \text{ mod } A,$$

for every A-set A on R_1. Here $K(A)$ may depend on A. If we can find

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$K(A)$ independent of A, then T is a quasiconformal mapping, and vice versa [3]. Therefore the class of all Royden's maps includes the class of all quasiconformal mappings. Furthermore, the inclusion is always proper for every pair of R_1 and R_2.

3. **A relation to Royden's compactification.** The reason we call mappings with property (2) Royden's maps is clarified by the following:

THEOREM 1. *A Royden's map T of R_1 onto R_2 can be continued to a unique topological mapping T^* of R_1^* onto R_2^*. Conversely, a topological mapping T^* of R_1^* onto R_2^* always maps R_1 onto R_2 and $T=T^*|R_1$ is a Royden's map of R_1 onto R_2.*

The first assertion is known for quasiconformal mappings [4]. We may summarize Theorem 1 as follows: the topological structure of R^* characterizes the quasiconformal structure of R at the ideal boundary.

4. **Boundary behavior.** The topological extension T^* of a Royden's map T of R_1 onto R_2 gives a topological mapping of Γ_1 onto Γ_2, where $\Gamma=R^*-R$ is the *Royden's boundary* of R. Let Δ be the *Royden's harmonic boundary* of R, i.e., the totality of regular points in Γ with respect to the Dirichlet problem [7]. Then

THEOREM 2. *The topological extension T^* of a Royden's map T of R_1 onto R_2 gives a topological mapping of Δ_1 onto Δ_2.*

The properties $R\in O_G$, O_{HD}, or O_{HD}^n are all characterized by the set theoretic properties of Δ [6]. Therefore, as a corollary of Theorem 2, we conclude that each of the classes O_G, O_{HD}, and O_{HD}^n is preserved under Royden's maps. This assertion generalizes theorems of Pfluger [8] and Royden [9].

5. **The case for the half plane.** Let T be a Royden's map of the upper half plane $U=\{z|\operatorname{Im}(z)>0\}$ onto itself. We also denote $\partial U=\{z|\operatorname{Im}(z)=0\}$ and $\overline{U}=\{z|\operatorname{Im}(z)\geqq 0\}$.

THEOREM 3. *The map T can be continued to a topological mapping $\overline{T}$ of $\overline{U}$ onto $\overline{U}$.*

This is, of course, well known for quasiconformal mappings [1]. Clearly T is both directly and indirectly conformally invariant. Hence we may assume that $\overline{T}|\partial U$ is a monotone increasing topological mapping of $(-\infty, \infty)$ onto itself. Then

THEOREM 4. *There exists a constant $\rho\geqq 1$ such that*

$$\rho^{-1} \leqq \frac{\overline{T}(x+t) - \overline{T}(x)}{\overline{T}(x) - \overline{T}(x-t)} \leqq \rho \tag{3}$$

for any $x \in \partial U$ *and* $t > 0$.

The inequality (3) is often referred to as the ρ-condition. The validity of (3) is well known for quasiconformal mappings [2].

From the features of Royden's maps listed above we conclude that Royden's maps are topological mappings which are "quasiconformal at the ideal boundaries."

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University of California, Los Angeles and
Nagoya University