CENTRAL IDEMPOTENTS IN GROUP ALGEBRAS

BY DANIEL RIDER[1]

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1. **Introduction.** The group algebra $L^1(G)$ of a group G consists of all complex functions f on G which have finite norm:

$$\|f\| = \sum_{x \in G} |f(x)| < \infty.$$

Multiplication is given by convolution:

$$f*g(x) = \sum_{y \in G} f(xy^{-1})g(y).$$

A function $f \in L^1(G)$ is an *idempotent* if $f*f=f$; it is *central* if $f(xy) = f(yx)$ for all $x, y \in G$. The *support* of f is the set of x for which $f(x) \neq 0$. The *support group* of f is the subgroup generated by the support of f.

The idempotents on abelian groups have been completely characterized [**1**], [**2**]; in particular they have finite support groups. Rudin [**3**] gives examples of noncentral idempotents on nonabelian groups which have infinite support and, *a fortiori*, infinite support groups.

The purpose of this paper is to answer affirmatively the question raised [**3**], [**4**] as to whether or not central idempotents have finite support groups.

THEOREM. *The support group of a central idempotent is finite.*

2. **Proof of the theorem.** Let f be a central idempotent on G. We can assume that G is the support group of f.

If G' is the commutator subgroup of G then it follows [**4**, Theorem 2.2] that G' has finite index in G. We will show that Z = the center of G also has finite index in G. But this implies that G' is finite (see, for example, [**5**, Theorem 15.1.13]) so that G must also be finite.

Let $S = \{x_1, x_2, \cdots\}$ be the support of f. Since f is central and has finite norm, $\{gx_ig^{-1}: g \in G\}$ is finite for each x_i so that each x_i commutes with the elements of a subgroup of finite index. Let H_n be the normal subgroup generated by $x_1, x_2, \cdots, x_n$ and let Z_n be the ele-

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ments of G which commute with the elements of H_n. Z_n is normal and since H_n is generated by finitely many elements of S it follows that Z_n has finite index.

Choose n_1 so large that

$$\sum_{x \notin H_{n_1}} |f(x)| < 1/2 \tag{1}$$

and let $H = H_{n_1}$. We divide the remainder of the proof into two cases.

Case I. Assume $HZ_n = G$ for all $n \geqq n_1$. Let $x \in Z_{n_1}$. Then x commutes with the elements of H. Now $x \in H_n$ for all but finitely many n so that x commutes with the elements of Z_n for some $n \geqq n_1$. Hence $x \in Z$ so that $Z = Z_{n_1}$ and thus Z has finite index.

Case II. Assume $HZ_n \neq G$ for some $n \geqq n_1$. Now HZ_n is a proper normal subgroup of G of finite index. Thus there is a finite, nontrivial, irreducible unitary representation T of G which is constant on HZ_n. We will show that this is not possible.

Let U be any irreducible representation of G into the unitary operators on some Hilbert space K. U gives a representation of $L^1(G)$ and since f is central and idempotent

$$Uf = \sum_{x \in G} f(x^{-1}) U(x) = \delta(U) E$$

where E is the identity operator on K and $\delta(U) = 1$ or 0.

Let K_0 be the (finite dimensional) Hilbert space on which the $T(x)$ $(x \in G)$ act. $T(x) = I$, the identity operator, for $x \in H$.

Consider the tensor product of U and T acting on f:

$$\begin{aligned}(U \otimes T)f &= \sum_{x \in G} f(x^{-1})(U(x) \otimes T(x)) \\ &= \sum_{x \in G} f(x^{-1})(U(x) \otimes I) \\ &\quad + \sum_{x \notin H} f(x^{-1})(U(x) \otimes T(x) - U(x) \otimes I).\end{aligned}$$

Thus, since $U(x)$ and $T(x)$ have unit norm,

$$\|(U \otimes T)f - \delta(U)(E \otimes I)\| \leqq \sum_{x \notin H} |f(x^{-1})| \cdot 2.$$

Since $(U \otimes T)f$ is idempotent and the norm of the difference of distinct commuting idempotents is at least 1 it follows from (1) that

$$(U \otimes T)f = \delta(U)(E \otimes I) = Uf \otimes I. \tag{2}$$

Let $e_1, e_2, \cdots, e_d$ be an orthonormal basis for K_0 and let X be the character afforded by T:

$$X(x) = \sum_{t=1}^{d} \langle T(x)e_t, e_t\rangle \qquad (x \in G).$$

Let $f^*(x) = f(x)X(x^{-1})$, $f^* \in L^1(G)$. If $a, b \in K$ then it follows from (2) that

$$\begin{aligned}\langle Uf^*a, b\rangle &= \sum_{x\in G} f(x^{-1}) \sum_{t=1}^{d} \langle T(x)e_t, e_t\rangle\langle U(x)a, b\rangle \\ &= \sum_{t=1}^{d}\sum_{x\in G} f(x^{-1})\langle (T(x) \otimes U(x))a \otimes e_t, b \otimes e_t\rangle \\ &= \sum_{t=1}^{d} \langle (T \otimes U)fa \otimes e_t, b \otimes e_t\rangle \\ &= d\langle Ufa, b\rangle.\end{aligned}$$

Hence $Uf^* = dUf = Udf$ for every irreducible representation of G so that $f^*(x) = df(x)$ for $x \in G$ .Now this implies that $X(x^{-1}) = d$, and hence that $T(x) = I$, for $x \in S$, the support of f. Thus $T(x) = I$ for all $x \in G$ which is a contradiction.

References

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Massachusetts Institute of Technology