ON NONSEPARABLE REFLEXIVE BANACH SPACES

BY JORAM LINDENSTRAUSS[1]

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The purpose of this paper is to show that certain known results concerning separable spaces hold also for nonseparable reflexive Banach spaces. Our main result (Theorem 1) proves a special case of a conjecture of H. H. Corson and the author [1] while the corollary proves some conjectures of V. Klee (see for example [2]). In order to state Theorem 1 we introduce the following notation: Let Γ be a set; by $c_0(\Gamma)$ we denote the Banach space of scalar valued functions f on Γ, such that $\{\gamma; |f(\gamma)| > \epsilon\}$ is finite for every $\epsilon > 0$, with the sup norm.

THEOREM 1. *Let X be a reflexive Banach space. Then there is a one to one bounded linear operator from X into $c_0(\Gamma)$ for a suitable set Γ.*

This theorem was proved in [3] for spaces X which have the metric approximation property (M.A.P.) introduced by Grothendieck. We shall show here how to modify the proof in [3] so that it will not depend on the assumption concerning the M.A.P. As noted in [3] the following corollary is an easy consequence of Theorem 1 and known results.

COROLLARY 1. *Let X be a reflexive Banach space. Then*
(i) *X has an equivalent strictly convex norm.*
(ii) *X has an equivalent smooth norm.*
(iii) *The norm of X is Gateaux differentiable at a dense subset of X.*
(iv) *If K is a bounded closed convex subset of X then K is the closed convex hull of its exposed points.*

We pass to the proof of Theorem 1. It is clearly enough to consider only real spaces. Our first lemma holds for a general Banach space.

LEMMA 1. *Let X be a Banach space and let B be a finite-dimensional subspace of X. Let k be an integer and let $\epsilon > 0$. Then there is a finite-dimensional subspace Z of X containing B such that for every subspace Y of X containing B with* $\dim Y/B = k$ *there is a linear operator $T: Y \to Z$ with $\|T\| \leq 1+\epsilon$ and $Tb = b$ for every $b \in B$.*

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PROOF. Let P be a bounded linear projection from X onto B and let $U=(I-P)X$. Let M be a positive number such that

$$(1) \qquad (M+k)/(M-k) < 1+\epsilon, \qquad k^2(2k+3)\|I-P\| < \epsilon M.$$

Let $\{b_\mu\}_{\mu=1}^m$ be a finite set in B such that for every $b\in B$ with $\|b\|\leq M$ there is a μ such that $\|b-b_\mu\|<M^{-1}$. Let $\sum_k$ be the sphere of radius 1 in the k dimensional space R^k, that is $\sum_k=\{\lambda=(\lambda_1,\cdots,\lambda_k);\ \sum_{i=1}^k \lambda_i^2=1\}$. Let $\{\lambda^j\}_{j=1}^p$ be a finite subset of $\sum_k$ such that for every $\lambda\in\sum_k$ there is a j with $\|\lambda^j-\lambda\|<M^{-1}$ (in R^k we take the norm $(\sum\lambda_i^2)^{1/2}$). Let $S_U=\{u;\ u\in U,\ \|u\|\leq 1\}$ and let

$$f: \overbrace{S_U\times S_U\times\cdots\times S_U}^{k} \to R^{mp}$$

be defined by

$$f_{\mu,j}(u_1,\cdots,u_k)=\left\|\sum_{i=1}^k \lambda_i^j u_i+b_\mu\right\|, \quad 1\leq\mu\leq m, \quad 1\leq j\leq p.$$

We choose now qk elements $\{u_i^\gamma\}$, $1\leq\gamma\leq q$, $1\leq i\leq k$, in S_U such that for every $(u_1,u_2,\cdots,u_k)\in S_U\times S_U\times\cdots\times S_U$ there is a γ such that

$$(2) \qquad |f_{\mu,j}(u_1,\cdots,u_k)-f_{\mu,j}(u_1^\gamma,\cdots,u_k^\gamma)| < M^{-1}, \qquad 1\leq\mu\leq m, \quad 1\leq j\leq p.$$

Let Z be the subspace of X spanned by B and the $\{u_i^\gamma\}$, $1\leq\gamma\leq q$, $1\leq i\leq k$. We claim that this subspace has the required properties.

Take any $Y\subset X$ with $Y\supset B$ and $\dim Y/B=k$. Then there are vectors $\{u_i\}_{i=1}^k$ in U such that $\|u_i\|=1$ for every i,

$$(3) \qquad \left\|\sum_{i=1}^k \lambda_i u_i\right\| \geq \left(\sum_{i=1}^k \lambda_i^2\right)^{1/2} \Big/ k^2 \quad \text{for every choice of real } \{\lambda_i\}_{i=1}^k$$

and $Y=\operatorname{span}\{B,u_1,u_2,\cdots,u_k\}$. Let now γ be such that (2) holds for these $u_1,\cdots,u_k$. Define T: $Y\to Z$ by $T(\sum\lambda_i u_i+b)=\sum\lambda_i u_i^\gamma+b$. We claim that T has the required properties, that is, that for every $\lambda=(\lambda_1,\cdots,\lambda_k)\in\sum_k$ and every $b\in B$

$$(4) \qquad \left\|\sum_{i=1}^k \lambda_i u_i^\gamma+b\right\| \leq (1+\epsilon)\left\|\sum_{i=1}^k \lambda_i u_i+b\right\|.$$

Assume first that $\|b\|>M$. Then $\|\sum\lambda_i u_i^\gamma+b\|\leq\|b\|+k$ and $\|\sum_{i=1}^k\lambda_i u_i+b\|\geq\|b\|-k$ and (4) follows from the first inequality of (1).

Assume next that $\|b\| \leqq M$ and let μ and j be such that $\|b_\mu - b\| < M^{-1}$ and $\|\lambda^j - \lambda\| < M^{-1}$. Then

$$(5)\qquad \left\| \sum_{i=1}^{k} \lambda_i u_i^\gamma + b \right\| \leqq f_{\mu,j}(u_1^\gamma, \cdots, u_k^\gamma) + (k+1)/M$$

and

$$(6)\qquad \left\| \sum_{i=1}^{k} \lambda_i u_i + b \right\| \geqq f_{\mu,j}(u_1, \cdots u_k) - (k+1)/M.$$

Therefore, by (2),

$$(7)\qquad \left\| \sum_{i=1}^{k} \lambda_i u_i^\gamma + b \right\| \leqq \left\| \sum_{i=1}^{k} \lambda_i u_i + b \right\| + (2k+3)/M.$$

Also, by (3)

$$(8)\qquad 1 = \sum_{i=1}^{k} \lambda_i^2 \leqq k^2 \left\| \sum_{i=1}^{k} \lambda_i u_i \right\| \leqq k^2 \|I - P\| \left\| \sum_{i=1}^{k} \lambda_i u_i + b \right\| \cdot$$

The inequality (3) follows now from (7), (8), and the second inequality of (1). This concludes the proof of the lemma.

LEMMA 2. *Let X be a reflexive Banach space and let B be a finite dimensional subspace of X. Then there exists a linear operator $T: X \to X$ such that $\|T\| = 1$, the range of T is separable and $Tb = b$ for $b \in B$.*

PROOF. Let $Z_n \supset B$, $n = 1, 2, \cdots$, be subspaces of X given by Lemma 1 for $k = n$ and $\epsilon = n^{-1}$. Let Z be the subspace of X spanned by $\bigcup_{n=1}^{\infty} Z_n$. Let E be a finite-dimensional subspace of X containing B with $\dim E/B = n$. Then there is a linear operator $T_E: E \to Z$ such that $T_E b = b$, $b \in B$, and $\|T_E\| \leqq 1 + n^{-1}$. We extend T_E to a map (not linear) $\tilde{T}_E: X \to Z$ by putting $\tilde{T}_E x = 0$ for $x \in X \sim E$. In the space of maps from X to Z we take the topology of pointwise convergence and Z is taken in the w topology. By Tychonoff's theorem the net $\{\tilde{T}_E\}$ (the spaces E are ordered by inclusion) has a subnet converging to a map $T: X \to Z$. It is easily verified that T is linear, $\|T\| = 1$ and $Tb = b$ for every $b \in B$.

From Lemma 2 we easily get (see the proof of Lemma 1 in [3]) that the following stronger version of it is true.

LEMMA 3. *Let X be a reflexive Banach space, let $\{x_i\}_{i=1}^n$ be a finite set in X, let $\{f_j\}_{j=1}^m$ be a finite set in X^* and let $\epsilon > 0$. Then there is a linear operator $T: X \to X$ with $\|T\| = 1$ such that $Tx_i = x_i$ for every i, $\|T^* f_j - f_j\| < \epsilon$ for every j, and TX is separable.*

We are now ready to prove

PROPOSITION 1. *Let X be a reflexive Banach space and let Y and Z be separable subspaces of X and X^* respectively. Then there is a separable subspace W of X containing Y and a linear projection P from X onto W such that $\|P\|=1$ and $P^*f=f$ for every $f\in Z$.*

PROOF. Let $\{f_j\}_{j=1}^{\infty}$ be a dense subset of Z. By Lemma 3 we can construct, inductively, a sequence $\{Y^n\}_{n=1}^{\infty}$ of separable subspaces of X and a sequence T^n of linear operators $T^n: X\rightarrow Y^n$ such that

$$\|T^n\| = 1, \quad n=1, 2, \cdots \tag{9}$$

$$\|T^{n*}f_j - f_j\| \leqq n^{-1}, \quad 1 \leqq j \leqq n, \quad n = 1, 2, \cdots \tag{10}$$

(11) $T^n x_i^k = x_i^k$ for $1\leqq i\leqq n,\ 0\leqq k\leqq n-1$, and $n=1, 2, \cdots$ where $\{x_i^k\}_{i=1}^{\infty}$ is a dense subset of Y^k and $Y^0 = Y$.

It is easily verified that $W=\operatorname{span} \bigcup_{n=0}^{\infty} Y^n$ and $P=$ the limit of a convergent subnet $\{T^n\}_{n=1}^{\infty}$ have the required properties.

The proof that Proposition 1 implies Theorem 1 is given in [3]. The M.A.P. is used in [3] only in order to prove Proposition 1.

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