EXTENSIONS OF COMMUTING ISOTONE FUNCTIONS[1]

BY FRANK W. OWENS

Communicated by D. Scott, June 6, 1966

The following problem was suggested as a research problem by Ralph De Marr in Bull. Amer. Math. Soc. **70** (1964), 501:

Let A be a nonempty subset of the unit interval I. Let $f_0, g_0: A \to A$ be isotone functions (i.e., $f_0(x) \leqq f_0(y)$ if $x \leqq y$) such that $f_0(g_0(x)) = g_0(f_0(x))$ for all $x \in A$. Can f_0 and g_0 be extended to isotone functions $f, g: I \to I$ which still commute?

We shall show that the answer is yes under certain additional assumptions, and give a counterexample to the problem in the above form.

DEFINITION. $A \subset I$ is called left (right)-closed if any decreasing (increasing) sequence in A has a limit in A. We write $A^L(A^R)$ for the left (right)-closure of A.

REMARK. A is closed iff A is left-closed and right-closed, i.e., $\bar{A} = A^L \cup A^R$.

THEOREM 1. *If $A \cup \{\inf A\}$ is left-closed or $A \cup \{\sup A\}$ is right-closed, there exist commuting isotone extensions.*

PROOF. We give the proof for the case $A \cup \{\inf A\}$ is left-closed. The case $A \cup \{\sup A\}$ is right-closed is similar. Extend f_0 and g_0 to $[0, \inf A] \cup A$ by defining them to be zero on $[0, \inf A]$ if $\inf A \notin A$, and to be their respective values at $\inf A$ if $\inf A \in A$. Next extend f_0 and g_0 to $B = [0, \inf A] \cup A \cup [\sup A, 1]$ by defining them to be one on $[\sup A, 1]$ if $\sup A \notin A$, and to be their respective values at $\sup A$ if $\sup A \in A$. Define $j: I \to B$ by $j(x) = \inf\{y \in B \mid x \leqq y\}$. j is isotone on I, and $j \mid B =$ the identity function on B. The required extensions are $f = f_0 j$ and $g = g_0 j$. f and g are isotone since the composition of two isotone functions is isotone. $f_0 j \mid A = f_0$ and $g_0 j \mid A = g_0$. f and g commute on I since $f_0 j g_0 j = f_0 g_0 j = g_0 j_0 j = g_0 j f_0 j$.

Note. The proof of the case $A \cup \{\sup A\}$ is right-closed is the same except that we define $j: I \to B$ by $j(x) = \sup\{y \in B \mid y \leqq x\}$.

DEFINITION. Let $h: A \to A$ be isotone. Define $h^L: A^L \to A^L$ and $h^R: A^R \to A^R$ by

$$\begin{aligned} h^L(x) &= h(x), & x &\in A \\ &= \inf\{h(y) \mid x \leqq y \in A\}, & x &\in A^L - A \end{aligned}$$

[1] Partially supported by N. S. F. Grant GP-5855.

and

$$h^R(x) = h(x), \qquad x \in A$$
$$= \sup \{h(y) \mid x \geqq y \in A\}, \qquad x \in A^R - A.$$

REMARK. h^L and h^R are well defined. h^L is right continuous on A^L if h is right continuous on A, and h^R is left continuous on A^R if h is left continuous on A.

THEOREM 2. *If f_0 and g_0 are both right or left continuous on A, there exist commuting isotone extensions.*

OUTLINE OF PROOF. f_0^L and g_0^L (f_0^R and g_0^R) commute on $A^L(A^R)$ if f_0 and g_0 are right (left) continuous on A. Apply Theorem 1 to get f_0^Lj and g_0^Lj (f_0^Rj and g_0^Rj) for the required extensions.

These and similar theorems can be generalized to complete lattices.

We shall now give a counterexample to the problem in its weak form. Let $A = [0, 1/2) \cup (1/2, 1]$. Define f_0 and g_0 by

$$\begin{aligned} f_0(x) &= 3/4 && \text{for} \quad 0 \leqq x \leqq 3/8 \\ &= 4x/3 + 1/4 && \text{for} \quad 3/8 \leqq x < 1/2 \\ &= 11/12 && \text{for} \quad 1/2 < x \leqq 3/4 \\ &= 1 && \text{for} \quad 3/4 < x \leqq 1, \end{aligned}$$

and

$$\begin{aligned} g_0(x) &= 3/4 && \text{for} \quad 0 \leqq x < 1/2 \\ &= x + 1/4 && \text{for} \quad 1/2 < x \leqq 2/3 \\ &= 11/12 && \text{for} \quad 2/3 \leqq x < 11/12 \\ &= 1 && \text{for} \quad 11/12 \leqq x \leqq 1. \end{aligned}$$

f_0, g_0: $A \to [3/4, 1] \subset A$ are isotone functions which commute on A ($f_0(g_0(x)) = g_0(f_0(x)) = 11/12$ for $0 \leqq x < 1/2$, and $f_0(g_0(x)) = g_0(f_0(x)) = 1$ for $1/2 < x \leqq 1$). The unique isotone extensions f and g of f_0 and g_0 are defined at $x = 1/2$ by $f(1/2) = 11/12$ and $g(1/2) = 3/4$. But $f(g(1/2)) = f(3/4) = 11/12$ and $g(f(1/2)) = g(11/12) = 1$.

STANFORD UNIVERSITY