THE K THEORY OF THE PROJECTIVE UNITARY GROUPS

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Let $U=U(p^r)$ be the unitary group on complex p^r space, p an odd prime. Let $S^1\subset U$ be the set of matrices λI where λ is a complex number with $|\lambda|=1$ and I is the identity matrix. Then S^1 is the center of U and $PU(p^r)=PU=U/S^1$.

We determine the complex K^* groups for the spaces PU by first determining the mod qK^* groups of these spaces [2] then using the mod p Bockstein spectral sequence to obtain the p torsion. $K^*[PU(p^r)]$ and $H^*[PU(p^r)]$ have no q torsion for $q\neq p$ and the mod p Bockstein spectral sequences for these two groups are isomorphic; thus,

THEOREM 5.5. *$H^*[PU(p^r), Z]$ and $K^*[PU(p^r)]$ are isomorphic as abelian groups.*

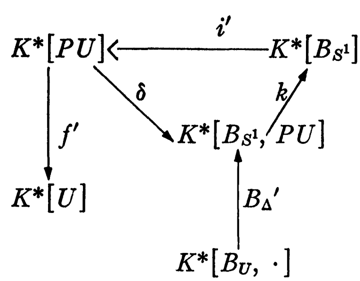
The details of these proofs will be published elsewhere. The outline follows:

Let B_{S^1} and $B_U=B_U(p^r)$ be the classifying spaces of the indicated groups. There are the following maps

$$U\xrightarrow{f}PU\xrightarrow{i}B_{S^1}\xrightarrow{B_\Delta}B_U.$$

Either i or B_Δ may be considered fibrations. We use the following diagram

(1)



Let $\rho_q\colon K^*[\ ,Z]\to K^*[\ ,Z_q]$ be the reduction [2] and $\beta_K\colon K[\ ,Z_q]\to K[\ ,Z]$ the Bockstein. $\exists\sigma_1,\sigma_2,\cdots,\sigma_{p^r}\in K^*[B_U,\ \cdot]\ni K^*[B_U]=Z[[\sigma_1,\cdots,\sigma_{p^r}]], H^*[B_U]=Z[\bar\sigma_1,\cdots,\bar\sigma_{p^r}]$, $K^*[U]=E[s\sigma_1,\cdots,s\sigma_{p^r}]$, $K^*[B_{S^1}]=Z[[y]]$, $H^*[B_{S^1}]=Z[\bar y]$. ρ_p is onto for these groups and it will be convenient to use x for $\rho_p(x)$ when possible. $kB_\Delta'(\sigma_i)=C_{p^r,i}y^i$ and $B_\Delta^*(\bar\sigma_i)=C_{p^r,i}\bar y^i$.

Since $kB_\Delta'(\sigma_i)=C_{p^r,i}\bar{y}^i$, $kB_\Delta'(\sigma_i)=0$ mod p for $i<p^r$; hence it follows from (1) using Z_p coefficients that for $i<p^r\ \exists x_i\in K^*[PU, Z_p]$ such that $\delta x_i=B_\Delta'\rho_p(\sigma_i)$. Let J be the set of integers j such that $1<j<p^r$ and j is not a pth power. A set $Y=\{Y_j|j\in J\}\subset K^*[PU, Z_p]$ is defined. Let $X=\{x_{p^i}|i=0, 1, \cdots, r-1\}$, $w=i'(y)$ and $\Lambda=E[Y]\otimes Z_p[w]/w^{p^r}$.

THEOREM 2.6. *$K[PU, Z_p]=E[X]\otimes\Lambda$ as an algebra. Λ is in the image of ρ_p so that $\beta_K(\Lambda)=0$.*

PROOF. The E_2 term of the mod p spectral sequence arising from the fibration $U\to PU\to B_{S^1}$ is $E_2=E[s\sigma_1, \cdots, s\sigma_{p^r}]\otimes Z_p[\bar{y}]$. Since $\delta x_i=B_\Delta'\rho_p(\sigma_i)$ it readily follows that $f'(x_i)=s\sigma_i$ for $i<p^r\ \therefore d_j(s\sigma_i)=0$ for all j and $i<p^r$. Since $B_\Delta^*(\bar{\sigma}_{p^r})=\bar{y}^{p^r}$ it follows that $d_{2p^r}(s\sigma_{p^r})=\bar{y}^{p^r}$. This describes the spectral sequence and we find that there is a filtration of $K^*[PU, Z_p]$ whose associated graded module $E_0K^*[PU, Z_p]$ is $E[s\sigma_1, \cdots, s\sigma_{p^r-1}]\otimes Z_p[\bar{y}]/\bar{y}^{p^r}$.

It is easy to see that the $x_{p^i}\in K^*[PU, Z_p]$ represent $s\sigma_{p^i}$ in $E_0K^*[PU, Z_p]$. The y_j where chosen to represent the $s\sigma_j$ for $j\in J$. w represents $\bar{y}$. This information is sufficient to show that the obvious map of $E[X]\otimes\Lambda$ to $K^*[PU, Z_p]$ is an isomorphism of algebras.

The appropriate tool for relating the Bockstein β_H in ordinary cohomology theory to the Bockstein β_K in K theory is the Atiyah-Hirzebruch spectral sequence. It is convenient to use the approach [10] for obtaining this spectral sequence.

Let $\mathfrak{U}$ be the infinite unitary group. Spaces $B_{\mathfrak{U}}(2i+2)$ are inductively defined by killing the $2i$th homotopy group of $B_{\mathfrak{U}}(2i)$ for $i\geqq 1$ and $B_{\mathfrak{U}}(2)=B_{\mathfrak{U}}$. In particular there is a commutative diagram of fiber spaces

$$(2)\qquad \begin{array}{ccccc} K[Z, 2i]^{S^1} & \xrightarrow{h_i} & B_{\mathfrak{U}}(2i+2) & \xrightarrow{f_{i+1}} & B_{\mathfrak{U}}(2i) \\ \downarrow & & \downarrow \psi_{i+1} & & \downarrow g_i \\ K[Z, 2i]^{S^1} & \longrightarrow & PK[Z, 2i] & \longrightarrow & K[Z, 2i] \end{array}$$

where $K[Z, 2i]$ is the indicated Eilenberg MacLane space with path space $PK[Z, 2i]$. Define $D^{2j,q}[X, Z]=[X, B_{\mathfrak{U}}(2j)^{S^q}]$ $j\geqq 1$, $q=0, 1$. $E_1^{2j,q}[X, Z]=[X, K[Z, 2j]^{S^q}]$ $j\geqq 1$, $q=0, 1$. $D=\sum D^{2j,q}$, $E_1=\sum E_1^{2j,q}$. The maps in diagram (2) allow us to define an exact couple

$$(3)\qquad \begin{array}{ccc} D[X, Z] & \xrightarrow{i} & D[X, Z] \\ \nwarrow k & & j \swarrow \\ & E_1[X, Z] = \tilde{H}[X, Z]. & \end{array}$$

Define

$$D^{2j,q}[X, Z_p] = [X, B\mathfrak{U}(2j+2)^{Z(p)\wedge S^q}] \qquad i \geqq 0,\ q = 0, 1,$$
$$E_1^{2j,q}[X, Z_p] = [X, K[Z, 2i+2]^{Z(p)\wedge S^q}] \qquad i \geqq 0,\ q = 0, 1$$

where $Z(p) = E^2 \cup_p S^1$. As above we obtain an exact couple

$$\begin{array}{ccc} D[X, Z_p] & \xrightarrow{i_p} & D[X, Z_p] \\ & \nwarrow k_p \qquad j_p \swarrow & \\ & E_1[X, Z_p] = \tilde{H}[X, Z_p]. & \end{array} \tag{4}$$

Let $E_r[X, Z]$ be the spectral sequence associated with the couple (3) and $E_r[X, Z_p]$ the spectral sequence associated with the couple (4). The first converges to $\tilde{K}^*[X, Z]$ and the second to $\tilde{K}^*[X, Z_p]$. The map $S^1 \to Z(p)$ induces maps $B_{\mathfrak{U}}(2i)^{Z(p)\wedge S^q} \to B_U(2i)^{S^1\wedge S^q}$ and $K[Z, 2i]^{Z(p)\wedge S^1} \to K[Z, 2i]^{S^1+S^q}$ which in turn define a map of the exact couple (4) into the exact couple (3). β_r: $E_r[X, Z_p] \to E_r[X, Z]$ is the induced map of spectral sequences. We show that $\beta_1 = \beta_H$ and that $\beta_\infty = E_0(\beta_K)$: $E_0\tilde{K}^*[X, Z_p] \to E_0\tilde{K}^*[X, Z]$.

Theorem 2.6 implies that the spectral sequence $E_r[PU, Z_p] \Rightarrow \tilde{K}[PU, Z_p]$ collapses so that $H[PU, Z_p] = E_\infty[PU, Z_p] = E_0K^*[PU, Z_p]$ and $\beta_H = \beta_\infty$. Let $\bar{x}_{p^i}$, $\bar{w} \in H^*[PU, Z_p]$ be the elements represented in $E_0K^*[PU, Z_p]$ by x_{p^i}, $w \in K^*[PU, Z_p]$. Then $\beta_H(\bar{x}_{p^i}) = \lambda_{i,0}\bar{w}^{p^i}$ and the highest power of p dividing $\lambda_{i,0}$ is p^{r-i-1}. One observes that $\delta\beta_K(x_{p^i}) = 0$ so that $\beta_K(x_{p^i}) \in \operatorname{Im} i'$; thus $\beta_K(x_{p^i}) = \alpha_i w^{p^i} + \sum_{j>0} \lambda_{i,j} w^{p^i+j}$. It follows that $E_0(\beta_K)\bar{x}_{p^i} = \alpha_i \bar{w}^{p^i}$ in $E_0K^*[PU, Z]$. $E_0(\beta_K) = \beta_\infty = \beta_H$ implies that $\lambda_{i,0} = \alpha_i$.

We must find a new set of generators z_{p^i} of the algebra $K^*[PU, Z_p]$ to replace the x_{p^i} and so that $\beta_K(z_{p^i}) = \lambda_{i,0} w^{p^i}$. Precisely

THEOREM 4.4. *There is a subset $Z_0 = \{z_{p^i} \mid i = 0, 1, \cdots, r-1\}$ of $K^*[PU, Z_p]$ such that $K^*[PU, Z_p]$ is isomorphic to $E[Z_0] \otimes \Lambda$ as an algebra and such that $\beta_K(\Lambda) = 0$, $\beta_K(Z_{p^i}) = \lambda_{i,0} w^{p^i}$.*

Theorem 4.4 gives sufficient information to determine the mod p Bockstein spectral sequence of $K^*[PU, Z]$. Let U_k be the set of products $z_{p^{r-k+1}} w^{p^{(r-k+1)(p-1)}}$, $z_{p^{r-k+2}} w^{p^{(r-k+2)(p-1)}}$, $\cdots$, $z_{p^{r-1}} w^{p^{(r-1)(p-1)}}$ and $W_k \subset Z_0$ the set $\{z_1, z_p, \cdots, z_{p^{r-k-1}}\}$. In particular, $U_1 = 0$, $U_{r+1} = U_\infty$ and $W_r = 0$.

THEOREM 4.5. *The kth term of the Bockstein spectral sequence for $K^*[PU, Z]$ is*

(i) $E_k = E[Y] \otimes E[W_k] \otimes E[U_k] \otimes \{E[z_{p^{r-k}}] \otimes Z_p[w]/w^{p^{r-k+1}}\}$

(ii) β_k, *the kth differential, is a derivation*, $\beta_k=0$ in $E[Y]\otimes E[W_k]\otimes E[U_k]$ *and* $\beta_k(z_{p^{r-k}})=\lambda^1_{r-k,0}w^{p^{r-k}}$ *for* $\lambda^1_{r-k,0}=(1/p^{k-1})\lambda_{r-k,0}$.

Theorems 4.4 and 4.5 hold for the ordinary cohomology of PU. In particular, the mod p Bockstein spectral sequences for $K^*[PU, Z]$ and $H^*[PU, Z]$ are isomorphic.

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