ON A CERTAIN INVARIANT OF A LOCALLY COMPACT GROUP

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Group here always means a locally compact Hausdorff group, *subgroup* means a closed subgroup. Let G be a group, H a subgroup and G/H the locally compact homogeneous space of left cosets $\dot{x}=xH$. We denote by $\mathfrak{K}(G)$ $[\mathfrak{K}(G/H)]$ the family of all compact subsets of $G[G/H]$. The group G acts on G/H in a natural way. If $X\subset G$ and $Y\subset G/H$, write XY for the set of all elements $x\dot{y}$, $x\in X$, $\dot{y}\in Y$. Now assume that G/H has a nontrivial invariant positive measure $d\dot{x}$, e.g. the left invariant Haar measure, if H is normal. For a measurable set U in G/H let $|U|$ or $|U|_{G/H}$ be its measure. Then we define:

$$I(G/H) = \sup_{K\in\mathfrak{K}(G)} \inf_{\substack{U\in\mathfrak{K}(G/H)\\ |U|>0}} \frac{|KU|_{G/H}}{|U|_{G/H}}.$$

Evidently $1\leqq I(G/H)\leqq\infty$ and $1=I(G/H)$ if G/H is compact. Let E be the trivial subgroup of order one in G. We identify G and G/E.

For a positive Radon measure μ and a Borel function f on G, the convolution $\mu*f$ is defined as

$$\mu*f(x) = \int_G f(y^{-1}x)\,d\mu(y).$$

If $\mathfrak{F}$ is a set of Borel functions, let $\mu*\mathfrak{F}$ be the set of all $\mu*f$, $f\in\mathfrak{F}$ (if this set is well defined). For $1\leqq p\leqq\infty$, let $\mathfrak{L}^p(G)$ be the usual $\mathfrak{L}^p$-space of the group G. If μ is a positive bounded Radon measure, then $\mu*\mathfrak{L}^p(G)\subset\mathfrak{L}^p(G)$ for all $p\geqq 1$. In [2] I proved a partial converse of this fact, as follows.

Let $p>1$. If $\mu*\mathfrak{L}^p(G)\subset\mathfrak{L}^p(G)$ and $I(G)<\infty$, then μ is bounded.

As I pointed out in [2], this implies the following.

Let $p>1$. If $\mathfrak{L}^p(G)$ is closed under convolution and $I(G)<\infty$, then G is compact.

This latter statement, without the hypothesis $I(G)<\infty$, is the so called $\mathfrak{L}^p$-conjecture, stated and discussed by Żelazko, Rajagopalan and others [3], [4], [5], [6], [7].

The main result of this note is an inequality for $I(G)$, which implies the finiteness of $I(G)$ for a fairly large class of groups. Actually it reduces the problem of checking this finiteness to the case of simple Lie groups and finitely generated discrete groups.

THEOREM 1. *Let H be a subgroup of G. If the homogeneous space G/H has a nontrivial positive invariant measure, then*

$$I(G) \leqq I(G/H)I(H).$$

We start with some definitions and lemmas. Let dx, $d\xi$ and $d\dot{x}$ be the (left) invariant measures in G, H and G/H respectively. For $f \in \mathfrak{L}^1(G)$, $\dot{x} \in G/H$ let

$$\dot{f}(\dot{x}) = \int_H f(x\xi)\, d\xi.$$

The function $\dot{f}$ is well defined almost everywhere on G/H and belongs to $\mathfrak{L}^1(G/H)$; furthermore

$$\int_G f(x)\, dx = \int_{G/H} \left(\int_H f(x\xi)\, d\xi \right) d\dot{x} = \int_{G/H} \dot{f}(\dot{x})\, d\dot{x},$$

(see e.g. [1, §2, no. 5]).

The image of a set $X \subset G$ in G/H is denoted by $\dot{X}$, the characteristic function of a set A by χ_A.

LEMMA 1. *Let $K \in \mathfrak{K}(G)$, $W \in \mathfrak{K}(H)$ and $Q = K^{-1}K \cap H \in \mathfrak{K}(H)$. Then*

$$|KW|_G \leqq |\dot{K}|_{G/H} |QW|_H.$$

PROOF. For $\dot{x} \notin \dot{K}$, $x \in \dot{x}$ and $\xi \in H$ we have $x\xi \notin KW$, hence $\chi_{KW}(x\xi) = 0$, $\dot{\chi}_{KW}(\dot{x}) = 0$. For $x \in K$ we have

$$x^{-1}KW \cap H \subset K^{-1}KW \cap H = (K^{-1}K \cap H)W = QW,$$

hence $\chi_{KW}(x\xi) = \chi_{x^{-1}KW}(\xi) \leqq \chi_{QW}(\xi)$ and

$$\dot{\chi}_{KW}(\dot{x}) = \int_H \chi_{KW}(x\xi)\, d\xi \leqq \int_H \chi_{QW}(\xi)\, d\xi = |QW|_H.$$

Finally

$$|KW|_G = \int_G \chi_{KW}\, dx = \int_{G/H} \dot{\chi}_{KW}\, d\dot{x} = \int_K \dot{\chi}_{KW}\, d\dot{x} \leqq |QW|_H \int_{\dot{K}} d\dot{x}$$
$$= |\dot{K}|_{G/H}\, |QW|_H.$$

LEMMA 2. *For $K \in \mathfrak{K}(G)$ the following inequality holds:*

$$|\dot{K}|_{G/H} \leqq \inf_{W \in \mathfrak{K}(H)} \frac{|KW|_G}{|W|_H} \leqq |\dot{K}|_{G/H} I(H).$$

PROOF. The right part of the inequality is an immediate consequence from Lemma 1. Now let $x \in K$ and $W \in \mathfrak{K}(H)$. Then $W \subset x^{-1}KW$, hence for $\xi \in H$ we have:

$$\chi_{KW}(x\xi) = \chi_{x^{-1}KW}(\xi) \geqq \chi_W(\xi),$$

$$|KW|_G = \int_{\dot{K}}\int_H \chi_{KW}(x\xi)\, d\xi\, d\dot{x} \geqq \int_{\dot{K}}\int_H \chi_W(\xi)\, d\xi = |\dot{K}|_{G/H}|W|_H,$$

which proves the left part of the inequality.

Now we can prove Theorem 1. Take K, $U \in \mathfrak{K}(G)$ with $|U|_G > 0$. For $\epsilon > 0$, Lemma 2 implies the existence of a $W \in \mathfrak{K}(H)$ with $|W|_H > 0$ such that

$$|(KU)^{\cdot}|_{G/H} = |\dot{K}U|_{G/H} \leqq \frac{|KUW|_G}{|W|_H} \leqq |\dot{K}\dot{U}|_{G/H}(I(H) + \epsilon),$$

$$|\dot{U}|_{G/H} \leqq \frac{|UW|_G}{|W|_H}.$$

It follows that

$$\inf_{V \in \mathfrak{K}(G)} \frac{|KV|_G}{|V|_G} \leqq \frac{|KUW|_G}{|UW|_G} \leqq \frac{|\dot{K}U|_{G/H}}{|\dot{U}|_{G/H}}(I(H) + \epsilon).$$

Now $\mathfrak{K}(G/H) = \{\dot{U} \mid U \in \mathfrak{K}(G)\}$, so taking the infimum over $\dot{U}$ and then the supremum over $K \in \mathfrak{K}(G)$ we get $I(G) \leqq I(G/H)(I(H) + \epsilon)$. This proves Theorem 1.

COROLLARY 1. *Let $G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = E$ be a normal series of G. If $I(H_i/H_{i+1}) < \infty$ for $i = 0, \cdots, n-1$, then $I(G) < \infty$. If $I(H_i/H_{i+1}) = 1$ for $i = 0, \cdots, n-1$, then $I(G) = 1$.*

COROLLARY 2. *If G has a finite normal series with compact or abelian factors, then $I(G) = 1$.*

COROLLARY 3. *If G is as in Corollary 2, every p-admissible positive Radon measure is finite. If in this case, $\mathfrak{L}^p(G)$ is closed under convolution for some $p > 1$, then G is compact.*

In [2] a measure μ was called p-admissible, if $\mu * \mathfrak{L}^p \subset \mathfrak{L}^p$.

The next corollary states the main result of [5].

COROLLARY 4. *If G is solvable, and $\mathfrak{L}^p(G)$ is closed under convolution for some $p > 1$, then G is compact.*

The next two theorems are especially useful for discrete groups.

THEOREM 2. *If H is an open subgroup of G, then $I(H) \leqq I(G)$.*

PROOF. Let the Haar measures in G and H be normalized, so that $|X|_G = |X|_H$ for $X \subset H$. Let $\Delta(x)$ be the modular function of G. Then for M measurable in H and $x \in G$, we have: $|Mx|_G = \Delta(x)|M|_G = \Delta(x)|M|_H$. Now, if $U \in \mathfrak{K}(G)$, there exist $x_1, \cdots, x_r \in G$ and $U_1, \cdots, U_r \in \mathfrak{K}(H)$ such that $U = \bigcup_1^r U_i x_i$. For $K \in \mathfrak{K}(H)$ it follows that $KU = \bigcup_1^r KU_i x_i$, the union being disjoint. Hence

$$\frac{|KU|_G}{|U|_G} = \frac{\sum_{i=1}^r |KU_i x_i|_G}{\sum_{i=1}^r |U_i x_i|_G} = \frac{\sum_{i=1}^r \Delta(x_i)|KU_i|_H}{\sum_{i=1}^r \Delta(x_i)|U_i|_H}$$

$$\geqq \min_i \frac{\Delta(x_i)|KU_i|_H}{\Delta(x_i)|U_i|_H} \geqq \inf_{V \in \mathfrak{K}(H)} \frac{|KV|_H}{|V|_H},$$

and finally

$$I(G) \geqq \sup_{K \in \mathfrak{K}(H)} \inf_{U \in \mathfrak{K}(G)} \frac{|KU|_G}{|U|_G} \geqq \sup_{K \in \mathfrak{K}(H)} \inf_{V \in \mathfrak{K}(H)} \frac{|KV|_H}{|V|_H} = I(H).$$

THEOREM 3. *Let Ω be the system of all open, compactly generated subgroups of G. Then*

$$I(G) = \sup_{H \in \Omega} I(H).$$

PROOF. From Theorem 2 it follows that $J = \sup_\Omega I(H) \leqq I(G)$. If $K \in \mathfrak{K}(G)$, there exists $H \in \Omega$ with $K \subset H$ and

$$\inf_{\mathfrak{K}(G)} \frac{|KU|_G}{|U|_G} \leqq \inf_{\mathfrak{K}(H)} \frac{|KV|_H}{|V|_H} \leqq I(H) \leqq J,$$

and hence $I(G) \leqq J$, $I(G) = J$.

COROLLARY 5. *If G is abelian, then $I(G) = 1$.*

This was proved in [2]. It also follows easily from Theorems 1 and 3.

COROLLARY 6. *Let G be discrete and Φ the family of all finitely generated subgroups of G. Then $I(G) = \sup_\Phi I(H)$.*

COROLLARY 7. *Let G be discrete and Ψ a family of subgroups such that* (1) *for $A, B \in \Psi$ there exists $C \in \Psi$ with $A, B \subset C$,* (2)*$G \subset = \bigcup_{H \in \Psi} H$. Then $I(G) = \sup_\Psi I(H)$.*

This follows from the previous corollary.

COROLLARY 8. *$I(G)=1$ if G is discrete and locally finite.*

COROLLARY 9. *Suppose that G is discrete and contains a well-ordered chain Φ of subgroups H_μ such that:*

(1) $H_0=E=\{e\}$, e the identity in G;

(2) H_μ is normal in $H_{\mu+1}$;

(3) $H_\lambda=\bigcup_{\mu<\lambda} H_\mu$, if λ is a limit ordinal;

(4) $G=H_\omega$ for an ordinal ω;

(5) $H_{\mu+1}/H_\mu$ is either abelian or locally finite (more general: $I(H_{\mu+1}/H_\mu)=1$).

Then $I(G)=1$.

PROOF. We use induction on ω. If $\omega=0$, there is nothing to prove. Now assume Corollary 9 is true for ordinals less than ω. Then especially $I(H_\lambda)=1$ if $\lambda<\omega$, hence Corollary 7 gives $I(G)=1$ if ω is a limit ordinal. Otherwise $\omega=\tau+1$, $I(H_\tau)=1$, $I(G/H_\tau)=1$ and so $I(G)=1$, according to Corollary 1.

We conclude with some remarks and a list of open problems.

1. Do there exist groups G with $1<I(G)<\infty$? In [2] we mentioned that $I(G)=\infty$ if G is the discrete free group with countably many free generators. Theorem 2 tells us that $I(G)=\infty$ if G is discrete and contains a nonabelian free subgroup.

2. Let H be a subgroup of G such that G/H is compact, but eventually without having an invariant measure. Is it still true that $I(G)\leqq I(H)$, or at least that $I(G)<\infty$ if $I(H)<\infty$?

3. Let H be a discrete central subgroup of G. Is $I(G/H)=I(G)$?

So far as I can see, an affirmative answer to questions 2 and 3 would imply that every G contains an open normal subgroup H with $I(H)=1$. It seems likely that there are also connections between the problems discussed here and the existence of invariant means on $C(G)$ and $\mathfrak{L}^\infty(G)$.

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