COBORDISM OF GROUP ACTIONS

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Let G be a compact Lie group and M a compact G manifold without boundary, i.e. a C^∞ manifold with a differentiable action of G on M. M^n is said to be G-cobordant to zero $M \sim_G 0$ if there exists a compact G manifold Q^{n+1} with $\partial Q = M$. Note that in this case M_G (the fixed point set of M) $= \partial Q_G$. M_G and Q_G are both disjoint unions of closed submanifolds (of varying dimension) of M, Q respectively. Let $\nu(M_G, M)$ denote the normal bundle of M_G in M; $\nu(M_G, M) \to M_G$ is a G-vector bundle in the sense of [5]. A partial converse to the statement $\nu(M_G, M) = \partial \nu(Q_G, Q)$ is given by

PROPOSITION 1 ([2, p. 10]). *If $\nu(M_G, M)$ is cobordant to zero as a G-vector bundle, i.e. if there exists a manifold W and a G-vector bundle $E \to W$ with $\partial W = M_G$, $E | \partial W = \nu(M_G, M)$ then M is G-cobordant to a manifold M' with $M'_G = \varnothing$.*

PROOF. Form the manifold $M \times I \cup_f E(1)$ where $E(1)$ denotes the unit disc bundle in E and

$$f\colon E(1) \mid \partial W = \nu(M_G, M) \xrightarrow[\text{fix}]{\exp} M \times 1.$$

Then note that, after smoothing,

$$\begin{aligned} \partial(M \times I \cup_f E(1)) &= M \times 0 \cup (M \times 1 - f(E(1) \mid \partial W) \cup \partial E(1)) \\ &= M \times 0 \cup M'. \end{aligned}$$

Hence, one may view the G-cobordism class of $\nu(M_G, M)$ as a first obstruction to finding a cobordism $M \sim_G 0$. Higher obstructions are formulated in terms of a spectral sequence. For simplicity we deal only with the unoriented case.

Let V be an orthogonal representation of G and let V^n denote the n-fold direct sum of V with itself and $S(V)$ the unit sphere in V. Consider the category of manifolds $\mathcal{G}(V)$ where M is in $\mathcal{G}(V)$ iff M can be imbedded in $S(V^n)$ for some n. One can then define the cobordism groups $\mathfrak{N}_n(V) = \mathfrak{N}_n(\mathcal{G}(V))$ of n dimensional G-manifolds in $\mathcal{G}(V)$ (see [5]). It was shown in [5] that if G is finite or abelian then $\mathfrak{N}_n(V) \approx \pi_1^{V^{2n+3}}(T_k(V^{2n+3} \oplus R), \infty)$ where $\pi_1^{V^{2n+3}}(T_k(V^{2n+3} \oplus R), \infty)$ de-

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notes the equivariant homotopy classes of maps of $S(V^{2n+3}\oplus \mathbf{R})$ into $T_k(V^{2n+3}\oplus \mathbf{R})$ the Thom space of the universal bundle of k-planes in $V^{2n+3}\oplus \mathbf{R}$. Let f be such a map; then proposition 1 may be reinterpreted as saying

PROPOSITION 1′. *Any homotopy of*

$$f \mid S(V^{2n+3} \oplus \mathbf{R})_G : S(V^{2n+3} \oplus \mathbf{R})_G \to T_k(V^{2n+3} \oplus \mathbf{R})_G$$

may be covered by a homotopy of f.

It was shown in [5] that there are only a finite number of conjugacy classes of isotropy groups occurring in $\mathcal{G}(V)$; let $(H_1), \cdots, (H_r)$ denote the conjugacy classes ordered by $(H_i) < (H_j)$ iff there is a $g\in G$ with $gH_ig^{-1}\subset H_j$ but $gH_ig^{-1}\neq H_j$. Define the level $H_i > n$ if $H_i < H_j$ and level $H_j > n-1$; level $G=0$ by definition and level $H_i = n$ if level $H_i > n-1$ but not level $H_i > n$. We may filter $\mathcal{G}(V)$ by subcategories $\mathcal{G}^i(V)$ where M is in $\mathcal{G}^i(V)$ if for each $x\in M$ level $(G_x)\geqq i$ and G_x is the isotropy group of x. One then has the corresponding cobordism groups $D_{n,i} = \mathfrak{N}_n(\mathcal{G}^i(V))$. Let $D_{n,i} = D_{n,0}$ for $i\leqq 0$ and let $D^{n,i}$ denote the image of $D_{n,i}$ in $D_{n,0} = \mathfrak{N}_n(V)$. We define $E_{n,i}$ $n\geqq 0$, $i\geqq 0$, as the cobordism group of differentiable G-vector bundles $E\to M$ where M is a compact G-manifold and

(i) $\dim E = n$;
(ii) E is in $\mathcal{G}(V)$;
(iii) $S(E)$ is in $\mathcal{G}(V)^{i+1}$ where $S(E)$ is the unit sphere bundle in E;
(iv) level $(G_x) = i$ for all $x\in M$.

Define $E_{n,i} = 0$ for $i < 0$. Vector bundles with fibre dimension zero are included.

THEOREM. *There is a graded exact couple*

$$\begin{array}{ccc} D & \xrightarrow{w} & D \\ & \partial\nwarrow \quad \swarrow\nu & \\ & E & \end{array}$$

where

$$D = \sum_{n,i} D_{n,i}, \qquad E = \sum_{n,i} E_{n,i}$$

with

$$E^r_{n,i} \Rightarrow E^\infty_{n,i} = D^{n,i}/D^{n,i+1}.$$

In particular

$$\mathfrak{N}_n(V) \approx \sum_{i=0}^{\infty} E_{n,i}^{\infty}.$$

The maps are as follows: Define $w: D_{n,i} \to D_{n,i-1}$ by $w([M]) = [M]$; if M is in $\mathcal{G}^i(V)$ then M is in $\mathcal{G}^{i-1}(V)$. Define $\partial: E_{n,i} \to D_{n-1,i+1}$ by $\partial(E \to M) = S(E)$; ∂ is well defined by (i), (ii), and (iii). Define $\nu: D_{n,i} \to E_{n,i}$ by $\nu([M]) = [\nu(M_i, M)]$ where $M_i = \{x \in M \mid \text{level } (G_x) = i\}$; M_i is a closed submanifold since M is in $\mathcal{G}^i(V)$. Conditions (i)–(iv) are clearly satisfied. Exactness follows from straightforward geometric arguments.

The groups $E_{n,i}$ may be described as follows: let H be an isotropy group on level i and let W be an r dimensional representation of H with $W \subset V^s \mid H$ for some s where $V^s \mid H$ means V^s considered as an H space.

Let $P(H, W)$ be the group of $N(H)$ (normalizer of H in G) equivariant bundle maps of $W \times_H N(H)$ into itself which are diffeomorphisms on the base space $N(H)/H$. We have the exact sequence $0 \to O_H(W) \to P(H, W) \to N(H)/H \to 0$ where $O_H(W)$ is the group of H equivariant orthogonal transformations of W.

PROPOSITION 2. *$E_{n,i}$ is isomorphic to the direct sum of $\mathfrak{N}_t(BP(H, W))$ over all such representations of H and all conjugacy classes of subgroups on level i; $\mathfrak{N}_t(BP(H, W))$ denotes the ordinary cobordism group (see [1, p. 45]) of the classifying space of $P(H, W)$ and $t = n - \dim W - \dim G/H$.*

PROOF. Let $E \to M$ be a bundle in $E_{n,i}$ with $(G_x) = (H)$ for all $x \in M$. By equivariance, it suffices to consider the $N(H)$ bundle $E \mid M_H \to M_H (M_H = \{x \in M \mid G_x = H\})$ since $M = M_H \times_{N(H)} G$ ([3, p. 42]); but M_H is a $N(H)/H$ principal bundle over M/G and hence one can see that $E \mid M_H \to M_H \to M/G$ is an $N(H)$ fibre bundle with fibre $N(H) \times_H W$ and structural group $P(H, W)$ ([4, p. 40]). Any element of $E_{n,i}$ is the disjoint union of such bundles.

To describe the differential we let $K \subset H$ be an isotropy group on level $i+1$; then $W \mid K = W_0 \oplus W_1$ where K operates trivially on W_0. $S(W_0)$ is a $N(K, H)/K$ principal bundle where $N(K, H)$ denotes the normalizer of K in H. Form the $N(H)$ bundle U over $BP(H, W)$ with fibre $N(H) \times_H S(W)$, $U = E_P \times_P (N(H) \times_H S(W))$ where $E_P \to BP(H, W)$ is the universal principal bundle; then $U_{(K)}/N(H) = U(K)$ is a bundle over $BP(H, W)$ with fibre $S(W_0)/N(K, H)$ and there is a map $i: U(K) \to BP(K, W_1)$ which classifies the normal bundle of $U_{(K)}$ in U. Then for any $[M, f] \in \mathfrak{N}_t(BP(H, W))$ we have the diagram

$$\begin{array}{ccccc} f^*U(K) & \xrightarrow{f_*} & U(K) & \xrightarrow{i} & BP(K, W_1) \\ \downarrow & & \downarrow & & \\ M & \xrightarrow{f} & BP(H, W). & & \end{array}$$

Clearly $d([M, f]) = \sum [f^*U(K), \ i \circ f_*] \in E_{n-1,i+1}$ where $[f^*U(K), i \circ f_*] \in \mathfrak{N}_s(BP(K, W_1))$, $s = n-1-\dim W_1 - \mathrm{im}\ G/K$, and the sum extends over all conjugacy classes (K) on level $i+1$ with $K \subset H$.

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