ON THE MAXIMAL RING OF QUOTIENTS OF $C(X)$

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1. Let $Q(X)$ denote the maximal ring of quotients (in the sense of Johnson [4] and Utumi [5]) of the ring $C(X)$ of continuous real-valued functions on the completely regular Hausdorff space X. This ring has been studied by Fine, Gillman, and Lambek [1] and realized by them as the direct limit of the subrings $C(V)$, V a dense open subset of X (i.e., the union of these $C(V)$'s, modulo the obvious equivalence relation). From this representation of $Q(X)$, it follows that if X and Y have homeomorphic dense open subsets, then $Q(X)$ and $Q(Y)$ are isomorphic. The full converse to this is false (see below). In this note a proof of the following is described.

THEOREM 1. *Let X and Y be separable metric spaces. If $Q(X)$ and $Q(Y)$ are isomorphic, then X and Y have homeomorphic dense open subsets.*

In particular, the spaces R^n, $n=1, 2, \cdots$ ($R=$the reals) have pairwise nonisomorphic Q's, thus settling a question[1] raised in [1]. That $Q(R)$ is not isomorphic to $Q(R^n)$, for $n>1$, was shown by F. Rothberger and J. Fortin. (See [2], and [1, p. 16].)

The main purpose of this note is to present a fairly simple solution to this question, and therefore the possible generalizations of Theorem 1 will not be discussed here. These generalizations, and related questions, will be treated in detail in a later paper.

The proof of Theorem 1 will now be described.

2. Homomorphisms of $C(Y)$ into $C(X)$ are well understood [3, Chapter 10]. If $\tau: X\rightarrow Y$ is continuous, $\phi(f)=f\circ\tau$ defines a homomorphism $\phi: C(Y)\rightarrow C(X)$. Conversely, if Y is realcompact, and $\phi: C(Y)\rightarrow C(X)$ is a homomorphism with $\phi(1)=1$, then ϕ is induced by a continuous function in this manner.

Now, let W_0 be a dense open subset of X, and let $\tau: W_0\rightarrow Y$ be continuous and additionally satisfy: for each dense open subset V of Y, $\tau^{-1}[V]$ is dense in X. Then $\phi(f)=f\circ\tau$ defines a homomorphism $\phi: Q(Y)\rightarrow Q(X)$. Evidently, ϕ satisfies

(*) for each dense open subset V of Y, there is a dense open subset W of X such that $\phi[C(V)]\subset C(W)$.

[1] The author is indebted to Professor Nathan J. Fine for communicating this question, and for many valuable conversations concerning it.

Conversely, if Y is hereditarily realcompact, and $\phi: Q(Y)\to Q(X)$ is a homomorphism with $\phi(1)=1$, and satisfying (*), then ϕ is induced by a continuous function in the manner described.

(*) states that ϕ respects the direct limit representations for the Q's. A homomorphism satisfying (*) will be called a *dl-homomorphism*, and an isomorphism ϕ such that both ϕ and ϕ^{-1} satisfy (*), a *bi-dl-isomorphism*.

PROPOSITION 2. *Let X and Y be hereditarily realcompact. $Q(X)$ and $Q(Y)$ are isomorphic by a bi-dl-isomorphism iff X and Y have homeomorphic dense open subsets.*

The situation with X and βX is interesting. For V a dense open subset of βX, and $f\in C(V)$, define $\phi(f)=f| V\cap X$. A dl-isomorphism $\phi: Q(\beta X)\to Q(X)$ results. In [1] it is shown that each continuous function on a dense open subset of X is extendible to a continuous function on a dense open supset of βX; hence ϕ is onto $Q(X)$. ϕ^{-1} is a dl-isomorphism iff each dense open subset of X is C-embedded in some dense open subset of βX. Choose for X the rationals P. Like any realcompact space, P is C-embedded in no space in which P is dense, and ϕ^{-1} is not a dl-isomorphism. In fact, there is no dl-isomorphism of $Q(P)$ onto $Q(\beta P)$, for it can be shown that such a mapping would be induced by a homeomorphism of a dense open subset of βP onto a subset of P, and such homeomorphisms do not exist.

3. For $f, g\in Q(X)$, define $f\geqq g$ if $f(x)\geqq g(x)$ for all $x\in\text{dom} f\cap \text{dom } g$. $Q(X)$ is thus a partially ordered ring. $Q^*(X)$ (the subring of bounded functions) is a metric space under

$$\rho(f, g) = \sup\{ |f(x) - g(x)| : x \in \text{dom} f \cap \text{dom } g\}.$$

(For much more on these matters, see [1].)

LEMMA 3. *Let X be separable and first-countable. Let A be a lattice subring of $Q(X)$ which contains constants and is closed under bounded inversion (i.e., if $f\in A$ and $f\geqq 1$, then $1/f\in A$); let A^* be complete (metrically, under ρ). Then there is a dense open subset V of X with $A\subset C(V)$.*

The proof of this goes as follows. If, for every V, $A\not\subset C(V)$, then the "singularities" of the functions in A are dense in some open set G; hence, some countable subset $\{p_1, p_2, \cdots\}$ of these singularities is also dense in G. For each n, $f_n\in A^*$ can be found for which the oscillation of f_n at p_n is nonzero. Upon suitable choice of real numbers $a_1, a_2, \cdots$, the partial sums of $\sum_{n=1}^{\infty} a_n f_n$ form a Cauchy sequence in A^* with no limit in $Q(X)$.

Theorem 1 is easily proved from Lemma 3. Let X and Y be separable metric spaces, and let ϕ be an isomorphism of $Q(Y)$ onto $Q(X)$. By routine arguments (ϕ preserves order, etc.), for any dense open subset V of Y, $\phi[C(V)]$ satisfies the hypotheses of Lemma 3. Hence ϕ (and by the same reasoning, ϕ^{-1}) is a dl-isomorphism. Proposition 2 now applies.

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