AN ENTIRE TRANSCENDENTAL FUNCTION WHOSE INVERSE TAKES SETS OF FINITE MEASURE INTO SETS OF FINITE MEASURE

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Communicated by W. Fuchs, April 25, 1966

In a recent issue of this journal [2] the following research problem was posed by F. Gross:

Let S be an arbitrary region of finite measure.[1] Does there exist a transcendental meromorphic function with the property that the pre-image $f^{-1}(S)$ is of finite measure?

The following theorem answers the above question not just for meromorphic functions but for entire functions also.

THEOREM. *There exists an entire transcendental function $f(z)$ whose inverse takes sets of finite measure into sets of finite measure.*

PROOF. Let $D = \{z = x+iy \mid x > 0,\ |y| < 1/(1+x^2)\}$ and let E be the complement in the plane of the closure of D. Also let $c(z)$ be a conformal map from the outside of $|z-1| = \epsilon$ (ϵ a small positive constant to be determined later) into the unit disk which takes the point at infinity into the origin. By a particular case of Runge's theorem [3] and Rouche's theorem there exists a rational function $R_1(z)$ with the following properties:

(i) $R_1(z)$ has a pole only at $z = 2$.

(ii) $R_1(z)$ so closely approximates $c(z)$ on E that $R_1(z)$ on E has the following properties:

(a) measure $R_1(E)$ less than one.

(b) $R_1(z)+z$ is 1-1 (this is always possible by choosing ϵ small enough) on E.

(c) $\lambda = |R_1(-1) - R(-2)| \neq 0$.

By the same argument there exists a rational function $R_2(z)$ which has the following properties:

(i) $R_2(z)$ has a pole only at $z = 3$.

(ii) $R_2(z)$ so closely approximates $R_1(z)$ in $E \cup G_3 (= \{z = x+iy \mid x < 3/2\})$ that $R_2(z)$ on $E \cup G_3$ has the following properties:

(a) measure $R_2(E)$ less than one.

(b) $R_2(z)+z$ is 1-1 on E.

(c) $|R_2(z) - R_1(z)| < \lambda/8$ on $E \cup G_3$.

Continuing inductively there exists a rational function $R_n(z)$ which has the following properties:

[1] Dr. Gross was kind enough to point out to me that in the statement of the problem accidentally the words "of finite measure" were left out.

(i) $R_n(z)$ has a pole only at $z=n+1$.

(ii) $R_n(z)$ so closely approximates $R_{n-1}(z)$ in $E\cup G_{2n-1}$ $(=\{z=x+iy \mid x<(2n-1)/2\})$ that $R_n(z)$ on $E\cup G_{2n-1}$ has the following properties:

(a) measure $R_n(E)$ less than one.

(b) $R_n(z)+z$ is 1-1 on E.

(c) $|R_n(z)-R_{n-1}(z)| <\lambda/2^{n+1}$ on $E\cup G_{2n-1}$.

By a theorem of Ostrowski [1] on the convergence of meromorphic functions in the chordal metric $\lim_{n\to\infty} R_n(z)=h(z)$ exists and is meromorphic in the plane. However by the way we choose the poles of $R_n(z)$ to converge to infinity $h(z)$ is clearly analytic. Since measure of $h(E)$ is less than or equal to one $h(z)$ cannot be a nonconstant polynomial and $h(z)$ cannot be constant since $|h(-1)-h(-2)| \geqq \lambda/2$. Now let $f(z)=h(z)+z$ and let S be any set of finite measure in the plane. The part of $f^{-1}(S)$ contained in D certainly has finite measure. The part of $f^{-1}(S)$ in E (call it K) must have finite measure, otherwise we would be led to a contradiction since [measure of $f(K)$] $\geqq$ [measure of K] $-$ [measure of $h(K)$] (from $|f'(z)|^2 \geqq 1-|h'(z)|^2$ for all z in E and the fact $f(z)$ has to be 1-1 on E) and the measure of $h(K)$ is less than one.

References

1. C. Carathéodory, *Theory of functions*, Vol. 1, p. 190, Chelsea, New York, 1954.

2. F. Gross, *Function theory* 18, Bull. Amer. Math. Soc. **71** (1965), 853.

3. S. Saks and A. Zygmund, *Analytic functions*, p. 177, Monographie, Matematyczne, vol. 28, Warsaw and Wraclaw, 1952.

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