ON CENTRAL TOPOLOGICAL GROUPS

BY SIEGFRIED GROSSER[1] AND MARTIN MOSKOWITZ[2]

Communicated by E. Hewitt, April 25, 1966

Introduction. Let G be a locally compact group and Z its center. We shall be concerned with the class $[Z]$ of all locally compact groups G such that G/Z is compact. In studying $[Z]$-groups the chief goal we have in mind is to obtain a natural generalization of the theory of compact groups and that of locally compact abelian groups broad enough to be nontrivial and which enables one to extend virtually all the important results pertaining to the aforementioned theories; this is done for the structure and representation theory. (For the latter see "Representation Theory of Central Topological Groups," p. 831 of this Bulletin).

1. **Extension theorems.** An interesting and useful feature of $[Z]$-groups is the extendibility of certain vector-valued homomorphisms defined on subgroups to crossed homomorphisms on the whole group.

Let G be a locally compact group, ρ a continuous finite-dimensional real representation of G on V and N a closed normal subgroup of G. As usual, a continuous function $\psi: G \rightarrow V$ is called a crossed homomorphism if $\psi(g_1 g_2) = \rho(g_1)(\psi(g_2)) + \psi(g_1)$, for all g_1, g_2 in G, and a continuous function $\phi: N \rightarrow V$ is called G-invariant if $\phi(g)(\rho(x)) = \phi(g X g^{-1})$, for all x in N, g in G.

THEOREM 1.1. *Let G, N, ρ, V be as above and suppose $G \in [Z]$. If ρ is a G-invariant homomorphism on N mapping into V^N, the N-fixed part of V, and either N or Z is open in G then ϕ extends to a crossed homomorphism ψ on G.*

The proof of this theorem makes use of the extendibility of continuous vector-valued homomorphisms defined on subgroups of locally compact abelian groups as well as the following well-known lemma due to P. Cartier [1], [5].

LEMMA 1.2. *Let G, N, ρ, V be as above. If G/N is compact and ϕ is a G-invariant homomorphism on N taking values in V^N then ϕ extends to a crossed homomorphism $\psi: G \rightarrow V$.*

[1] Research partially supported by National Science Foundation grants GP-1610 and GP-3685.

[2] Research partially supported by National Science Foundation and Office of Army Research, Durham.

With the same methods one proves the following theorem.

THEOREM 1.3 (1) *Let G, N, ρ, V be as above and suppose $G\in[Z]$. If $N\subseteq Z(G)$ and ϕ: $N\to V$ is a G-invariant homomorphism then ϕ extends to a crossed homomorphism ψ on G.*

(2) *Let G, N, V be as above and suppose $G\in[Z]$. If N is abelian and ϕ: $N\to V$ is a continuous homomorphism satisfying $\phi(gXg^{-1})=\phi(x)$ for all x in N and g in G, then ϕ extends to a continuous homomorphism ψ on G.*

COROLLARY. *If N is a subgroup of $Z(G)$, where $G\in[Z]$ and ϕ: $N\to V$ is a continuous homomorphism then ϕ extends to a continuous homomorphism ψ on G.*

In contrast to the case where G is abelian, in general it is not possible to prove extension theorems analogous to the above corollary if the range of ϕ is not a vector group but a torus.

These extension theorems yield a direct estimate of the rank of the fundamental group of an analytic $[Z]$-group. This allows one to answer a question raised by P. Smith [**18**].

2. **Structure theorems.** We utilize the following terminology: $\mathfrak{A}$ denotes the group of all topological group automorphisms of the locally compact group G. (As is well-known, in the compact-open topology, $\mathfrak{A}$ is not, in general, a topological group; a finer topology must be employed ([**0**], [**5**])). The group $\mathfrak{I}$ of inner automorphisms of G is a normal, but not necessarily closed, subgroup of $\mathfrak{A}$. If $\mathfrak{B}$ is any subgroup of $\mathfrak{A}$ we say that G has small $\mathfrak{B}$-invariant neighborhoods of the identity if every neighborhood of 1 in G contains a $\mathfrak{B}$-invariant neighborhood of 1. If $\mathfrak{B}=\mathfrak{I}$, we simply say that G has small invariant neighborhoods of 1. The class of such groups will be denoted by [SIN]; they were first studied by G. Mostow [**13**] and K. Iwasawa [**7**].

For a study of the relationship between $[Z]$-groups and [SIN]-groups as well as for other purposes the following theorem is of importance.

THEOREM 2.1. (ASCOLI THEOREM FOR AUTOMORPHISM GROUPS). *A subgroup $\mathfrak{B}$ of $\mathfrak{A}$ has compact closure if and only if G has small $\mathfrak{B}$-invariant neighborhoods of* 1 *and the $\mathfrak{B}$-orbits of points of G have compact closure.*

The statement of the theorem points up the relevance of the generalization introduced above of the notion of small invariant neighborhoods of 1.

Closely related with $[Z]$ is the class $[FIA]^-$ of groups G with the property that $\mathfrak{J}$ has compact closure in $\mathfrak{A}$; this was introduced by R. Godement [**3**]. The three classes are related, as follows.

THEOREM 2.2. (1) *If* $G \in [Z]$ *then* $G \in [SIN]$ *and* $\mathfrak{J}(x)$, *the* $\mathfrak{J}$-*orbit of* x, *is compact, for every* x *in* G.

(2) $G \in [FIA]^-$ *if and only if* $G \in [SIN]$ *and* $(\mathfrak{J}(x))^-$ *is compact, for every* x *in* G.

The main structural result is

THEOREM 2.3 (STRUCTURE THEOREM FOR $[Z]$-GROUPS). *If* $G \in [Z]$ *then* $G = V \times H$ (*direct product*) *where* V *is a vector group and* H *contains a compact open normal subgroup.*

This generalizes the classical structure theorem for locally compact abelian groups [**19**]. The proof proceeds by considering, successively, analytic groups (Lie algebra theory), connected groups (Yamabe's approximation theorem [**2**] and the theorem of Freudenthal-Weil [**19**], [**5**], groups with compact component group (Iwasawa's theorem on the automorphism group of a compact group), general $[Z]$-groups (the extension theorems stated above). In the case of connected G the structure theorem is also an immediate consequence of facts concerning representations of $[Z]$-groups and the theorem of Freudenthal-Weil.

The following corollary of the theorem constitutes a complete generalization of a well-known theorem of discrete group theory [**14**].

COROLLARY 1. *If* $G \in [Z]$ *then the closure of the commutator subgroup of* G *is compact.*

COROLLARY 2. *A normal vector subgroup of a* $[Z]$-*group is a direct factor.*

Another consequence of the structure theorem together with facts mentioned above is

THEOREM 2.4. *If* $G \in [Z]$ *then* G *is a projective limit of Lie groups.*

With the help of these theorems and related facts it is possible to clarify the relationships between $[Z]$, $[FIA]^-$, $[SIN]$, and $[FD]^-$, the latter class consisting of locally compact groups G such that the closure of the commutator subgroup of G is compact. The results of this section together with [**13**] and [**7**] may be regarded as the clarification called for by A. Weil of the complex of questions raised on page 129 of [**19**].

3. Applications of the structure theorem. One of the more notable of these is the following. Denote by P the set of periodic elements of G, i.e., the elements of G that belong to compact subgroups.

THEOREM 3.1. *If $G \in [Z]$ then:*

(1) *P is a closed characteristic subgroup of G which contains every compact subgroup.*

(2) *P consists of those elements x of G which belong to compact normal subgroups.*

(3) *G/P has no periodic elements (except the identity); in fact, $G/P = V \times D$ (direct product) where V is a vector group and D is a discrete torsionfree abelian group.*

With the help of this theorem as well as Pontrjagin Duality a criterion can be given for a $[Z]$-group to have a (unique) maximum compact subgroup. For compactly generated $[Z]$-groups these results specialize to quite precise statements about their structure. Finally, the notion of division closure of subgroups of a discrete group can be generalized to topological groups and all the desirable properties of the division closure can be established, mainly with the help of Theorem 3.1. This gives rise to a closure operation in the sense of lattice theory on the power set of any $[Z]$-group.

The authors would like to express warmest thanks to Professor G. P. Hochschild for his careful criticism of the results and his generous help in the preparation of this paper.

REFERENCES

0. J. Braconnier, *Sur les groupes topologiques localment compact.* J. Math. Pures Appl. (9) **27** (1948), 1–85.

1. P. Cartier, Séminaire S. Lie, 1954/55–1955/56.

2. V. Gluskov, *The structure of locally compact groups and Hilbert's 5th problem*, Amer. Math. Soc. Transl. (2) **15** (1960), 55–95.

3. R. Godement, *Analyse harmonique dans les groupes centraux.* I. *Fonctions centrales et caractères*, C. R. Acad. Sci. Paris, **225** (1947), 19–21.

4. ———, Review of: F. I. Mautner, *Infinite-dimensional irreducible representations of certain groups*, Math. Rev. **12** (1951), 588.

5. G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965.

6. K. H. Hofmann and Paul Mostert, *Splitting in topological group*, Mem. Amer. Math. Soc. No. 43 (1963).

7. K. Iwasawa, *Topological groups with invariant compact neighbourhoods of the identity*, Ann. of Math. (2) **54** (1951), 345–348.

8. E. van Kampen, *Locally bicompact abelian groups and their character groups*, Ann. of Math. **36** (1935).

9. J. L. Kelley, *General topology*, van Nostrand, Princeton, N. J., 1952.

10. A. Malcev, *On the simple connectedness of invariant subgroups of Lie groups*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **34** (1942), 10–13.

11. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.

12. M. Moskowitz, *Locally compact abelian groups and homological algebra*, Ph.D. Thesis, University of California, Berkeley, 1964.

13. G. Mostow, *On an assertion of Weil*, Ann. of Math (2) **54** (1951), 339–344.

14. B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc. (3) **1** (1951), 178–187.

15. ———, *Groups with finite classes of conjugate subgroups*, Math. Z., 63–64, 1955–1956.

16. L. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, N. J., 1939.

17. ———, *Topologische Gruppen*, Teil I, II, B. G. Teubner Verlagsgesellschaft, Leipzig, 1957.

18. P. Smith, *On the fundamental group of a group manifold*, Ann. of Math. **36** (1935),

19. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann et Cie, Paris, 1953.

UNIVERSITY OF CALIFORNIA AT BERKELEY,
CORNELL UNIVERSITY AND
UNIVERSITY OF CHICAGO