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A PERTURBATION LEMMA

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1. **Introduction.** We will prove the following lemma and investigate some of its implications: namely, a short proof by Goldberg [1] of the basic perturbation theorem of Kato [2], avoiding previous homotopy arguments; an extension of results of Trotter and Nelson [3] for semigroup generators; and a criterion for well-posed perturbed problems in spaces that are not necessarily complete. For further references and more information, see [1], [2], and [3].

Throughout this paper all operators are linear with domains subspaces of a normed linear space X and ranges subspaces of a normed linear space Y. If an operator B perturbs an operator T, we assume that $D(B)\supset D(T)$.

In this section, the spaces need not be complete.

LEMMA 1. *Let T^{-1} and B be bounded operators with $\|B\| < \|T^{-1}\|^{-1}$. Then*

$$\dim Y/\mathrm{Cl}\,(R(T)) = \dim Y/\mathrm{Cl}\,(R(T+B)). \tag{1.1}$$

PROOF.[2] We use the known result (e.g., see [1] for a proof) that if $\|B\| < \|T^{-1}\|^{-1}$, then

$$\dim Y/\mathrm{Cl}\,(R(T+B)) \leqq \dim Y/\mathrm{Cl}\,(R(T)). \tag{1.2}$$

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[2] Concerning this little result, let $\|B\| <\alpha\|T^{-1}\|^{-1}$. The author appreciates discussions with Dr. Seymour Goldberg, who proved it for $\alpha=1/2$ in his lectures. The main trick in the proof can be seen for the case $\alpha=3/4$. The author also appreciates the aid of Mr J. Kuttler in extending the result from $\alpha=3/4$ to $\alpha=7/8$.

From (1.2) we see that the proof of (1.1) reduces to showing

$$\dim Y/\mathrm{Cl}\,(R(T)) \leqq \dim Y/\mathrm{Cl}\,(R(T+B)). \tag{1.3}$$

The trick in showing (1.3) is to perturb and unperturb T successively by fractions of B of just the right size, using (1.2) at each stage.

We first note that $\|B\| < \|T^{-1}\|^{-1}$ implies that there exists some integer $n>0$ such that $\|B\| < [(2^n-1)/2^n]\cdot\|T^{-1}\|^{-1}$. We now let $c_k=(2^{n-k})/(2^n-1)$ for $k=1, \cdots, n$. For convenience, we also let $c_0=0$. We now claim that

$$\begin{aligned}\dim Y \Big/ \mathrm{Cl}\left(R\left[T+\left(\sum_{k=0}^{m-1} c_k\right)B\right]\right)& \\ \leqq \dim Y \Big/ \mathrm{Cl}\left(R\left[T+\left(\sum_{k=0}^{m} c_k\right)B\right]\right)&\end{aligned} \tag{1.4}$$

for each $m=1, \cdots, n$. We use (1.2), with a perturbation by $-c_mB$, to show that (1.4) holds, since

$$\begin{aligned}\|-c_mB\| &< [(2^{n-m})/(2^n-1)]\cdot[(2^n-1)/2^n]\cdot\|T^{-1}\|^{-1} \\ &= 2^{-m}\|T^{-1}\|^{-1},\end{aligned} \tag{1.5}$$

and noting that $\sum_{k=0}^m c_k=[2^n/(2^n-1)]\cdot[(2^m-1)/2^m]$, we have for $x\in D(T)$

$$\begin{aligned}\frac{\left\|\left(T+\left(\sum_{k=0}^m c_k\right)B\right)x\right\|}{\|x\|} &\geqq \frac{\|Tx\|}{\|x\|}-\left(\sum_{k=0}^m c_k\right)\frac{\|Bx\|}{\|x\|} \\ &> [1-(2^m-1)/2^m]\cdot\|T^{-1}\|^{-1} \\ &= 2^{-m}\|T^{-1}\|^{-1}.\end{aligned} \tag{1.6}$$

From (1.5) and (1.6) we have $\|-c_mB\| < \|(T+(\sum_{k=0}^m c_k)B)^{-1}\|^{-1}$, which by (1.2) is sufficient for (1.4) to hold for each $m=1, \cdots, n$. Combining these n inequalities then yields (1.3).

2. **Perturbation theory.** In this section we assume that both X and Y are complete spaces. We first state some known definitions.

An operator T is called normally solvable (n.s.) if it is closed and has closed range. If the kernel of T is closed, the minimum modulus is defined by $\gamma(T)=\inf_{x\in D(T),x\notin N(T)}[\|Tx\|/d(x, N(T))]$. We also have the three indices $\alpha(T)=\dim N(T)$, $\beta(T)=\dim Y/R(T)$, and if either $\alpha<\infty$ or $\beta<\infty$, $\kappa(T)=\alpha(T)-\beta(T)$.

Goldberg [1] has employed Lemma 1 to give a short proof of a perturbation theorem of Kato [2]; roughly, that if T is n.s. and B is bounded and small, then $T+B$ is n.s. and certain relations hold between the indices of T and those of $T+B$. We now state for future reference the known extended version of this theorem, wherein B is only required to be T-bounded.

THEOREM (PERTURBATION). *Let T be n.s. and possess index κ. Let B satisfy*

$$\|Bx\| \leqq a\|Tx\| + b\|x\| \tag{2.1}$$

for all $x \in D(T)$, where $b+a\gamma(T) < \gamma(T)$. Then $T+B$ is n.s., $\alpha(T+B) \leqq \alpha(T)$, $\beta(T+B) \leqq \beta(T)$, and $\kappa(T+B) = \kappa(T)$.

To illustrate the role that Lemma 1 can play in such a context, we note that if T also possesses a bounded inverse and $\|B\| < \gamma(T) \equiv \|T^{-1}\|^{-1}$, Lemma 1 states that

$$\kappa(T) = -\beta(T) = -\beta(T+B) = \kappa(T+B). \tag{2.2}$$

3. **Semigroup generators.** We now prove the following:

THEOREM 2. *Let A be the infinitesimal generator of a contraction semigroup on the Banach space X, and let B be a dissipative operator with $D(B) \supset D(A)$. If there exist constants a and b, with $a < 1$, such that for all $\phi \in D(A)$,*

$$\|B\phi\| \leqq a\|A\phi\| + b\|\phi\|, \tag{3.1}$$

then $A+B$ is the infinitesimal generator of a contraction semigroup.

REMARK. The above result, as an extension of an earlier result by Trotter, is obtained by Nelson [3] under the additional condition that $a < \frac{1}{2}$. For definitions, references, and applications to the Schrödinger equation and semigroup generation, see [3]. For our purposes, we will use: (i) Nelson's form of the Hille-Yosida-Phillips Theorem, characterizing a densely defined operator T as the infinitesimal generator of a contraction semigroup if and only if it is dissipative and there exists λ_0 such that $\lambda > \lambda_0$ implies that λ is in the resolvent set of T; (ii) any dissipative operator T satisfies $\|(\lambda - T)\phi\| \geqq \lambda\|\phi\|$ for all $\phi \in D(T)$ and all real λ; and (iii) the dissipative operators form a convex cone.

PROOF OF THEOREM 2. By the above remark, $A+B$ is dissipative and $(\lambda - A - B)^{-1}$ is continuous for any positive λ. The remainder of the proof thus consists of showing the existence of some λ_0 such that for $\lambda > \lambda_0$, $R(\lambda - A - B) = X$.

We will first show that for $a<\frac{1}{2}$, Theorem 2 is a direct corollary of the above-stated perturbation theorem. Then by a device motivated by Lemma 1, we will extend the result to $a<1$.

Suppose $a<\frac{1}{2}$. Then for any operator A and positive λ,

$$a\|A\phi\| + b\|\phi\| \leqq a\|(\lambda - A)\phi\| + (a\lambda + b)\|\phi\|. \tag{3.2}$$

Now let $\lambda>\lambda_0=\max\{\lambda_0(A), b/(1-2a)\}$. Then since A is dissipative,

$$\begin{aligned} b' + a\gamma(\lambda - A) &\equiv (a\lambda + b) + a\|(\lambda - A)^{-1}\|^{-1} \\ &< (1 - a)\lambda + a\|(\lambda - A)^{-1}\|^{-1} \\ &\leqq (1 - a)\|(\lambda - A)^{-1}\|^{-1} + a\|(\lambda - A)^{-1}\|^{-1} \\ &= \gamma(\lambda - A), \end{aligned} \tag{3.3}$$

which by the perturbation theorem and A's properties yields that

$$R(\lambda - A - B) = R(\lambda - A) = X. \tag{3.4}$$

Suppose $\frac{1}{2}\leqq a<1$. Then $a<(2^m-1)/2^m$ for some integer m. Let $\alpha=2^{m-1}/(2^m-1)$ and note that $\alpha a<\frac{1}{2}$. From (3.1) we have

$$\alpha\|B\phi\| \leqq \alpha a\|A\phi\| + \alpha b\|\phi\| \tag{3.5}$$

and thus $A+\alpha B$ is the infinitesimal generator of a contraction semigroup. From (3.1) we also have

$$\begin{aligned} \alpha\|B\phi\| \leqq \alpha a&\left\|\left(A + \alpha\left(\sum_{j=0}^{k-1} 2^{-j}\right)B\right)\phi\right\| \\ &+ \frac{1}{2}\alpha\left(\sum_{j=0}^{k-1} 2^{-j}\right)\|B\phi\| + \alpha b\|\phi\| \end{aligned} \tag{3.6}$$

for $k=1, \cdots, m-1$. Since $\sum_{j=0}^{k-1} 2^{-j}=(2^k-1)/(2^{k-1})$, (3.6) gives

$$2^{-k}\alpha\|B\phi\| \leqq \alpha a\left\|\left(A + \alpha\left(\sum_{j=0}^{k-1} 2^{-j}\right)B\right)\phi\right\| + \alpha b\|\phi\|, \tag{3.7}$$

which at each step yields $[A+\alpha(\sum_{j=0}^{k} 2^{-j})B]$ as an infinitesimal generator of a contraction semigroup, which for $k=m-1$ is the desired result for $A+B$.

COROLLARY 3. *Under the conditions of Theorem 2, except with $a<a_1$, $c(A+dB)$ is the infinitesimal generator of a contraction semigroup for all $c\geqq 0$ and all $0\leqq d\leqq 1/a_1$.*

4. **Well-posed problems.** Since Lemma 1 holds for spaces that are not necessarily complete, it would appear to be useful in other ways,

such as the following inference of well-posed (uniqueness, stability, existence) perturbations from well-posed base problems.

EXAMPLE 4. Let T^{-1} and B be bounded, $\|B\| < \|T^{-1}\|^{-1}$, and $R(T) = Y$. If $R(T+B)$ is closed, then the equation

$$(T + B)x = y \tag{4.1}$$

is well-posed.

PROOF. The uniqueness and stability follow from (1.6) when $m = n$ there. The existence of a solution follows from (1.1) and $R(T+B)$ closed, which imply that $R(T+B) = Y$.

Although the main feature of the example is that the spaces need not be complete, we may observe that if X is complete, the conclusion of the example is

$$R(T + B) \text{ closed} \Leftrightarrow T + B \text{ closed} \Leftrightarrow \text{well-posed (4.1)} \tag{4.2}$$

Furthermore, the hypotheses of the example imply that T is closed; hence if $D(B)$ is closed, it follows that $T+B$ is closed.

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