UNIFORMLY BOUNDED REPRESENTATIONS OF THE UNIVERSAL COVERING GROUP OF SL(2, *R*)[1]

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Communicated by E. Hewitt, November 23, 1965

In a recent paper, Pukanszky [6] has classified all the irreducible unitary representations of the universal covering group G of $G_1 = \mathrm{SL}(2, \boldsymbol{R})$. We construct analytic continuations of the "principal series" of representations of G into certain domains in the complex plane. The representations obtained by continuation are uniformly bounded and, in special cases, they are equivalent to those constructed for G_1 by Kunze and Stein in [4]. In the course of the investigation, several interesting connections with special functions occur. Full details and proofs will be published elsewhere.

I. **The principal series and complementary series.** For the purposes of analytic continuation, we need realizations of the irreducible unitary representations of G which are somewhat different than those constructed in [6]. The group G may be parametrized as follows. $G = \{(\gamma, \omega) \mid \gamma \in \boldsymbol{C}, |\gamma| < 1; \omega \in \boldsymbol{R}\}$ (see [1, p. 594]). The principal series of irreducible unitary representations of G may be realized in $L_2(T)$, T the unit circle in $\boldsymbol{C}$. This was originally suggested by Bargmann [1, p. 616]. The representations are indexed by two parameters h and s, $h \in \boldsymbol{R}$, $-\frac{1}{2} < h \leqq \frac{1}{2}$, $s \in i\boldsymbol{R}$ (pure imaginary). We exclude the pair $h = \frac{1}{2}$, $s = 0$, which gives a reducible representation. For $f \in L_2(T)$, $g = (\gamma, \omega) \in G$, the representation operators are given by

$$[U_h(g, s)f](e^{i\theta}) = e^{-2i\omega h}\left(\frac{1 + e^{i\theta}\bar{\gamma}}{1 + e^{-i\theta}\gamma}\right)^h \left| e^{i\theta}\bar{\gamma} + 1\right|^{-1-2s}(1 + |\gamma|^2)^{1/2+s} f(e^{i\theta}\cdot g), \tag{1}$$

where $e^{i\theta}\cdot g = e^{2i\omega}[(e^{i\theta} + \gamma)/(e^{i\theta}\bar{\gamma} + 1)]$ is the natural action of G on T. We remark that, for any pair $(h, s) \in \boldsymbol{C}^2$, the operator defined by (1) is a bounded operator on $L_2(T)$ and the map $g \rightarrow U_h(g, s)$ is a continuous representation of G on $L_2(T)$. The principal series of irreducible unitary representations of G_1 is obtained from (1) by setting $h = 0$ and $h = \frac{1}{2}$.

For any fixed h, $-\frac{1}{2} < h \leqq \frac{1}{2}$, and $s \in i\boldsymbol{R}$, the representations

[1] The results presented in this paper are a part of the author's Ph.D. thesis at Brandeis University, written under the direction of Professor R. A. Kunze.

[2] Partially supported by the National Science Foundation, Contract NSF GP-89.

$g \to U_h(g, s)$ and $g \to U_h(g, -s)$ are unitarily equivalent [6, p. 104]. We construct explicitly the family of intertwining operators and show that the intertwining property actually holds in a half-plane. These considerations lead naturally to the representation spaces for the complementary series of representations of G. (The existence of this series was not indicated in [1].)

If $f \in L_2(T)$ and $f(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} f_n e^{in\theta}$, the Fourier series of f, we set

$$(2) \qquad [A_h(s)f](e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} \lambda_{-n}(s, h) f_n e^{in\theta},$$

where $\lambda_n(s, h) = \Gamma(\frac{1}{2}+h+s)\Gamma(n+\frac{1}{2}+h-s)/\Gamma(\frac{1}{2}+h-s)\Gamma(n+\frac{1}{2}+h+s)$.

THEOREM 1. (a). *For* $-\frac{1}{2} < h \leqq \frac{1}{2}$, s *pure imaginary, the operator* $A_h(s)$ *defined by* (2) *is a unitary operator on* $L_2(T)$.

(b) *For any fixed* h, $-\frac{1}{2} < h \leqq \frac{1}{2}$, *and* $s \in \boldsymbol{C}$, $\operatorname{Re}(s) \geqq 0$, *the operator defined by* (2) *is a bounded operator on* $L_2(T)$ *and* $s \to A_h(s)$ *is an analytic operator-valued function on* $\operatorname{Re}(s) > 0$.

(c) *For any fixed* h, $-\frac{1}{2} < h \leqq \frac{1}{2}$, *and* $s \in \boldsymbol{C}$, $\operatorname{Re}(s) \geqq 0$,

$$A_h(s) U_h(g, s) = U_h(g, -s) A_h(s).$$

(d) *For* $-\frac{1}{2} < h < \frac{1}{2}$, $s \in \boldsymbol{R}$, $0 < s < \frac{1}{2} - |h|$, $A_h(s)$ *is a strictly positive operator on* $L_2(T)$.

The simplest method for proving the intertwining property (c) is to map the problem to another space. For $-\frac{1}{2} < h \leqq \frac{1}{2}$, $s \in i\boldsymbol{R}$, the operator

$$[E_h(s)f](x) = 2^s \pi^{-1/2} (1 - ix/1 + ix)^h \left| 1 - ix \right|^{-1-2s} f(x - i/-x - i)$$

is a unitary map from $L_2(T)$ to $L_2(\boldsymbol{R})$. Letting $\mathfrak{F}$ denote the Fourier transform on $\boldsymbol{R}$, we then consider the operator on $L_2(\boldsymbol{R})$ defined by

$$\hat{A_h}(s) = \mathfrak{F} E_h(-s) A_h(s) E_h(s)^{-1} \mathfrak{F}^{-1}.$$

LEMMA 1. *For* $f \in L_2(\boldsymbol{R})$,

$$[\hat{A_h}(s)f](t) = [\Gamma(\tfrac{1}{2} - h \operatorname{sgn} t + s)/\Gamma(\tfrac{1}{2} - h \operatorname{sgn} t - s)] \left| t \right|^{-2s} f(t).$$

The intertwining operators for G_1 (the case $h = 0, \frac{1}{2}$) were determined in this form by Kunze and Stein [4] (also see [3, p. 62]).

If we set

$$(3) \qquad \hat{U_h}(g, s) = \mathfrak{F} E_h(s) U_h(g, s) E_h(s)^{-1} \mathfrak{F}^{-1},$$

then part (c) of Theorem 1 is equivalent to $\hat{A_h}(s)\hat{U_h}(g, s) = \hat{U_h}(g, -s)\hat{A_h}(2s)$. For certain generators of G, the operators defined by (3) are easily computed and the intertwining property may be verified immediately.

For $-\frac{1}{2}<h<\frac{1}{2}$, $s\in\boldsymbol{R}$, $0<s<\frac{1}{2}-|h|$, and $f, g\in L_2(T)$, we set

$$(f, g)_s = (A_h(s)f, g). \tag{4}$$

By Theorem 1(d), (4) defines a positive definite hermitian form on $L_2(T)$. We complete $L_2(T)$ to a Hilbert space $H_{s,h}$ with respect to $(\cdot,\cdot)_s$.

THEOREM 2. *Suppose* $-\frac{1}{2}<h<\frac{1}{2}$, $s\in\boldsymbol{R}$, $0<s<\frac{1}{2}-|h|$. *Let* $U_h(g, s)$ *be the bounded operator on* $L_2(T)$ *defined by* (1). *Then*

(a) $U_h(g, s)$ *is a unitary operator on* $L_2(T)$ *considered as a subspace of* $H_{s,h}$ *and can be extended to a unitary operator, also denoted* $U_h(g, s)$, *on* $H_{s,h}$.

(b) *The map* $g\rightarrow U_h(g, s)$ *defines a continuous irreducible unitary representation of* G *on* $H_{s,h}$.

We note that part (a) is an immediate consequence of Theorem 1(c) and the fact that $U_h(g, s)^* = U_{\bar{h}}(g^{-1}, -\bar{s})$. This series of representations is called the *complementary series of G*.

II. **Analytic continuation of the principal series.** For any fixed h, $-\frac{1}{2}<h\leqq\frac{1}{2}$, let $\mathfrak{D}_h$ be the region defined as follows

$$\mathfrak{D}_h = \{s = \sigma + i\tau \mid -\tfrac{1}{2} < \sigma < \tfrac{1}{2}, \tau \neq 0 \text{ or } -\tfrac{1}{2} + |h| < \sigma < \tfrac{1}{2} - |h|, \tau = 0\}.$$

THEOREM 3. *There exist representations* $g\rightarrow R_h(g, s)$ *of* G *on* $L_2(T)$ *with the following properties*

(a) *For each fixed* h, $-\frac{1}{2}<h\leqq\frac{1}{2}$, *and* $s\in\mathfrak{D}_h$, $g\rightarrow R_h(g, s)$ *is a continuous, uniformly bounded representation of* G *on* $L_2(T)$.

(b) *The map* $s\rightarrow R_h(g, s)$ *is an analytic, operator-valued function on each* $\mathfrak{D}_h$, $-\frac{1}{2}<h\leqq\frac{1}{2}$.

(c) *The representations* $g\rightarrow R_h(g, s)$, $-\frac{1}{2}<h\leqq\frac{1}{2}$, $s\in i\boldsymbol{R}$, *are unitarily equivalent to the corresponding representations of the principal series.*

(d) *The representations* $g\rightarrow R_h(g, \sigma)$, $-\frac{1}{2}<h<\frac{1}{2}$, $0<\sigma<\frac{1}{2}-|h|$, *are unitarily equivalent to the corresponding representations of the complementary series.*

The basic idea in obtaining the operators $R_h(g, s)$ is to "normalize" the principal series as in [4] and [5]. We thus construct operators

$$R_h(g, s) = W_h(s)U_h(g, s)W_h(s)^{-1}, \qquad -\tfrac{1}{2} < h \leqq \tfrac{1}{2},$$

s pure imaginary, where $W_h(s)$ is a unitary operator on $L_2(T)$. The operators $R_h(g, s)$ have the properties that $R_h(g, s)=R_h(g, -s)$ and that $R_h(g, s)=R_h(g, 0)$ when g is restricted to certain subgroups of G. The first of these properties facilitates the proving of (c) and (d) in Theorem 3. The second property reduces the problem of analytic continuation to the continuation of one operator, $R_h(p, s)$ $p=(0, -\pi/2)\in G$.

The normalizing operator $W_h(s)$ is essentially a product of the form (convolution) (multiplication) (convolution). If we move to $L_2(\boldsymbol{R})$, the operator $\hat{W_h}(s)=\mathfrak{F}E_h(0)W_h(s)E_h(s)^{-1}\mathfrak{F}^{-1}$, which is unitarily equivalent to $W_h(s)$, has the form

$$[\hat{W_h}(s)f](t) = [\Gamma(\tfrac{1}{2} - h \operatorname{sgn} t + s)/\Gamma(\tfrac{1}{2} - h \operatorname{sgn} t - s)]^{1/2} | t |^{-s} f(t),$$

where $f\in L_2(\boldsymbol{R})$, that is, $\hat{W_h}(s)$ is the "square root" of $\hat{A_h}(s)$. We then consider the operator

$$\hat{R_h}(p, s) = \hat{W_h}(s)\hat{U_h}(p, s)\hat{W_h}(s)^{-1} = \mathfrak{F}E_h(0)R_h(p, s)E_h(0)^{-1}\mathfrak{F}^{-1}.$$

It is clear that a continuation of $\hat{R_h}(p, s)$ in the parameter s affords a continuation of $R_h(p, s)$.

If f is a compactly supported C^∞ function on $\boldsymbol{R}$, we have

$$\begin{aligned} &[\hat{R_h}(p, s)f](t) \\ &= [e^{ih\pi}/\sin(2s\pi)]\Big\{[\cos(h+s)\pi]\int_0^{\infty \operatorname{sgn} t} f(u)J_{-2s(\operatorname{sgn} t)}(2(|ut|)^{1/2})\,du \\ &\quad - [\cos(h-s)\pi]\int_0^{\infty \operatorname{sgn} t} f(u)J_{2s(\operatorname{sgn} t)}(2(|ut|)^{1/2})\,du\Big\} \\ &\quad + (2e^{ih\pi}/\pi)[\cos(h+s)\pi]^{1/2}[\cos(h-s)\pi]^{1/2}(\operatorname{sgn} t) \\ &\quad \cdot \int_{-\infty \operatorname{sgn} t}^{0} f(u)K_{2s}(2(|ut|)^{1/2})\,du, \end{aligned} \tag{5}$$

where J_ν and K_ν are the classical Bessel functions. For $h=0, \frac{1}{2}$, the operators defined by (5) are similar to those obtained in [**3**, p. 60] for SL(2, K), K a locally compact, totally disconnected, nondiscrete field.

Proceeding one step further, we compute $\mathfrak{M}\hat{R_h}(p, s)$, where $\mathfrak{M}$ denotes the (two-sided) Mellin transform on $\boldsymbol{R}$. For simplicity, we take $f\in C^\infty(0, \infty)$. Using formulas in [**2**, p. 49, 51], we get

$$\begin{aligned} [\mathfrak{M}\hat{R_h}(p, s)f](\alpha) &= \mathfrak{F}(1-\alpha)(e^{ih\pi}/\pi)\{[\cos(\alpha+h)\pi] \\ &\quad + [\cos(h+s)\pi]^{1/2}[\cos(h-s)\pi]^{1/2}\}\Gamma(\alpha-s)\Gamma(\alpha+s), \end{aligned} \tag{6}$$

where $\mathfrak{F}(1-\alpha)=\int_0^\infty f(u)u^{-\alpha}du$, $\alpha=\frac{1}{2}+i\beta$, $\beta\in\boldsymbol{R}$. With the form (6) the analytic continuation into the prescribed domain is a simple matter.

There is another series of representations of G indexed by a real parameter h, $|h|>0$, (corresponding to the discrete series of G_1). These representations may be realized in reproducing kernel spaces of holomorphic (or conjugate holomorphic) functions on the unit disk in $\mathbf{C}$. These representations can be normalized so that they have properties similar to the operators $R_h(g, s)$. In this form they can be continued into the entire complex plane minus the imaginary axis. The representations obtained by continuation are not uniformly bounded in this case.

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