WIENER'S WORK IN PROBABILITY THEORY

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In 1918 and 1919 Daniell's papers on what is now called the Daniell integral appeared. As an application he constructed the most general finite measure of Borel sets in Euclidean space of countable dimensionality. In 1933 Kolmogorov, in his formalization of probability as measure theory, rediscovered the Daniell result in constructing measures on Euclidean space of arbitrary dimensionality. After 1933 probability to a mathematician was no longer merely an extra-mathematical source of interesting problems in analysis with colorful interpretations, but was now a normal part of mathematics whose historical development was commemorated by the use of peculiar names (random variable, expectation, . . .) for commonplace mathematical concepts (measurable function, integral, . . .).

Wiener was an exception in his probability work in that he almost never used the classical nomenclature, and in fact usually even avoided using the standard classical results and conventions, some of which would have simplified and clarified his work. He came into probability from analysis and made no concession. If his work had been less forbiddingly formal, it might have had even more influence.

In a series of papers beginning with **[12]*** Wiener undertook a mathematical analysis of Brownian motion. It was accepted that Brownian paths were governed by probabilistic laws, and it seemed plausible that the paths were continuous. The problem was to construct and analyze a rigorous mathematical model. More than a decade before Kolmogorov's formalization of probability Wiener constructed a mathematical model of Brownian motion in which the basic probabilities were the values of a measure defined on subsets of a space of continuous functions. This measure has since been commonly called "Wiener measure." Fixing an origin in time and a direction in space, let $x(t)$ be the component in the specified direction of the displacement by time t of a Brownian particle. Then $x(0)=0$. For technical reasons it was advantageous to restrict t to a compact interval. Thus Wiener was led to consider the space C of continuous functions on $[0, 1]$, vanishing at 0, and to define a measure of subsets of C (based on the Daniell integral). The probability of any property of the displacement function was associated with the measure of the subset of C having this property. The Wiener measure of subsets of C

* The bold-faced numbers in brackets refer to the numbered references in the Bibliography of Norbert Wiener.

has the following properties: (Here if ω is a member of C, $X(t, \omega)$ is the value of ω at t.)

(a) The measure of C is 1.

(b) The function $X(t, \cdot)$ is a random variable (measurable function) on C and $X(t_2, \cdot)-X(t_1, \cdot)$ has a distribution which is Gaussian with mean 0 and mean square value $\alpha|t_2-t_1|$, where α is an assigned strictly positive parameter.

(c) If $t_0 < \cdots < t_n$, then $X(t_1, \cdot) - X(t_0, \cdot), \cdots, X(t_n, \cdot) - X(t_{n-1}, \cdot)$ are mutually independent.

In view of the definition of C the values of t here are restricted to the interval $[0, 1]$ but a simple transformation then yields a measure on the space of continuous functions on $[0, \infty)$, or $(-\infty, \infty)$ if desired, vanishing at 0, with properties (a), (b), (c) for the relevant parameter values. In his early work, and later also in his joint book **[92]** with Paley, Wiener studied the regularity of Brownian paths, proving that almost no function in C is of bounded variation in any interval, and finding estimates of the moduli of continuity for the functions.

It is characteristic of his writing that in the details of his construction Wiener leaves it to the reader to recognize that property (c) is true, as an implication of the explicit distribution formulas he writes down. In general he preferred writing formulas to making descriptive remarks except in his later years. This preference helped the rigor but not the reader.

In **[92]** $X(t, \omega)$ is defined in a different and very elegant way; $X(t, \cdot)$ is a function on the unit interval with Lebesgue measure, properties (a), (b), (c) hold, and $X(\cdot, \omega)$ is a continuous function for almost all ω. The construction does not involve the Daniell integral, and $X(t, \omega)$ is represented explicitly as the sum of a Fourier series with random coefficients.

In a little-known paper of 1921 **[18]**, Wiener applied his measure to obtain a second model for Brownian motion, a model which is the more exact model rediscovered by Ornstein and Uhlenbeck in 1930. In this model the paths are defined by functions which (almost all) have continuous first derivatives but these derivatives have infinite variation on every interval.

The stochastic integral $I(f)=\int f(t)d_tX(t, \omega)$ was one of Wiener's most fruitful ideas. (We shall write d for d_t below.) Here $X(t, \omega)$ is from the Brownian motion process, first model, and f is Lebesgue measurable and square integrable on $(-\infty, \infty)$, $f \in L_2(-\infty, \infty)$. The integral, a random variable, can be defined in a reasonable way even though the function $X(\cdot, \omega)$ has infinite variation on every interval, for almost all ω. The transformation $f \to I(f)$ is a linear L_2-norm preserving transformation from $L_2(-\infty, \infty)$ into the L_2 space of square integrable random variables. The stochastic integral, which was generalized later by Ito to allow integrands f which may depend on ω as well as t, was for Wiener and still remains a fundamental tool in a variety of contexts, only a few of which will be mentioned below.

In [92] it is shown that Wiener's harmonic analysis of functions on $(-\infty, \infty)$ is applicable to functions defined by the stochastic integral $\int f(x+iy+t)dX(t, \omega)$, where f is a function analytic in the strip $|y| < c$ and satisfies certain boundedness conditions. Moreover the asymptotic properties of the zeros of h are derived.

Suppose that $\{Y(t), -\infty < t < \infty\}$ is a stationary stochastic process in the sense that $Y(t)$ is a random variable and that the inner product $R(t) = (Y(t+s), \overline{Y}(s))$ does not depend on s. Then the function $R(\cdot)$, supposed continuous, is the Fourier transform of a measure, the spectral measure of the process, which dominates the harmonic analysis of the process. Wiener showed, and applied repeatedly, that the Brownian motion process acts as though the derivative $X'(t, \omega)$ process exists, is stationary, and has a constant spectral density. For example, if A is a suitably restricted linear operator the equation

$$A(Y) = Z(t, \omega) \tag{1}$$

can be solved to give a stationary $Y(t)$ process depending on the $Z(t)$ process, supposed given and stationary. The spectral measure dF_2 of the $Y(t)$ process is obtained from that of the $Z(t)$ process, dF_1, by means of a function g determined by A:

$$dF_2 = |g|^2 dF_1. \tag{2}$$

In many contexts (1) is replaced by

$$A(Y) = X'(t, \omega) \tag{1'}$$

where $X'(t, \omega)$ is from the fictitious derived process of the Brownian motion process and the equation is given meaning by a formal integration. In this way the solution of (1′) finally appears as a stochastic integral defining a stationary process and (2) is replaced by $dF_2 = \text{const}\, |g|^2 dt$ in accordance with Wiener's principle. Equation (1′) arises for example in noise problems in which the right side represents "white noise," a "Brownian driver" a source which to a first approximation produces power uniformly distributed over all frequencies.

The preceding theory has been stated in linear terms. Wiener also discussed analogous problems in a nonlinear context [108], [191] in which he used multiple stochastic integrals as a fundamental tool. The translational shift $X(t, \omega) \to X(t+s, \omega)$ defines a probability preserving transformation of Wiener measure (here the t-interval is $(-\infty, \infty)$ and the condition $X(0, \omega)=0$ is not imposed because only differences like $X(t_2, \omega)-X(t_1, \omega)$ are involved). Hence a multiple stochastic integral

$$\int \cdots \int f(t - t_1, \cdots, t - t_n)\, dX(t_1, \omega) \cdots dX(t_n, \omega) = h(t, \omega)$$

defines a family of random variables which is strictly stationary in the sense that joint distributions of sets of the $h(t, \cdot)$ are invariant under the shift. Wiener showed [**191**] how to evaluate moments of such multiple stochastic integrals and make a spectral analysis of the stationary process so defined. Moreover [**108**] he showed how families of this type "polynomial chaoses" could be used to approximate very general stationary processes. Ergodic theory is a natural tool in this area.

In summary, and neglecting Wiener's work in prediction theory which will be discussed elsewhere, Wiener's most important contributions to probability theory were centered about the Brownian motion process, now sometimes called the "Wiener process." He constructed this process rigorously more than a decade before probabilists had made their subject respectable and he applied the process both inside and outside mathematics in many important problems.

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