THE GALOIS THEORY OF INFINITE PURELY INSEPARABLE EXTENSIONS

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Communicated by A. D. Mostow, June 20, 1965

Introduction. Given a field K of characteristic $p\neq 0$, denote by $\mathrm{Der}(K)$ the set of all derivations of K. Then $\mathrm{Der}(K)$ is a vector space over K, and a Lie subring of the ring of additive endomorphisms of K. Moreover, $\mathrm{Der}(K)$ is closed under pth powers. A Lie ring satisfying this additional closure property is called a *restricted* Lie ring. Take any subfield F of K such that K over F is of exponent one, i.e., $K^p\subset F$. Denote by $\mathrm{Der}(K/F)$ the set of all derivations of K which vanish on F. Then $\mathrm{Der}(K/F)$ is a vector subspace and restricted Lie subring of $\mathrm{Der}(K)$. On the other hand, take a restricted Lie subring D of $\mathrm{Der}(K)$ which is also a vector subspace over K. Let $\Phi(D) = \{x\in K \mid \lambda(x)=0 \text{ for every } \lambda\in D\}$. Then $\Phi(D)$ is a subfield of K such that K over $\Phi(D)$ is of exponent one. This gives a one-to-one correspondence between subfields F of K over which K is *finite* and of exponent one, and restricted Lie subrings of *finite* dimension over K (cf. [1] and [2]). The purpose of this note is to extend this Galois correspondence to the infinite dimensional case. The first half of the correspondence is valid regardless of the dimension of K over F, i.e., $\Phi(\mathrm{Der}(K/F))=F$ if $K^p\subset F$ [1, p. 183]. However, to establish the second half of the correspondence, one must put a stronger condition on the vector subspace of $\mathrm{Der}(K)$, namely, that of p-convexity.

p-convexity. Let us fix a field K of characteristic $p\neq 0$. Since we shall only consider subfields F for which $K^p\subset F$, we should *designate* K^p as our base field. For every $x\in K$, let H_x denote the set of all λ in $\mathrm{Der}(K)$ such that $\lambda(x)=0$. H_x may be regarded as a "distinguished" hyperplane in $\mathrm{Der}(K)$. We call a subspace V of $\mathrm{Der}(K)$ *p-convex* if $V=\bigcap(V+H_x)$, the intersection being taken over all $x\in K$.

THEOREM 1. *Let V be a vector subspace of* $\mathrm{Der}(K)$ *which is p-convex, and let $F=\Phi(V)$. Then* $\mathrm{Der}(K/F)=V$, *which implies that every p-convex subspace of* $\mathrm{Der}(K)$ *is automatically a restricted Lie subring of* $\mathrm{Der}(K)$. *Conversely, if F is a subfield of K containing K^p, then* $\mathrm{Der}(K/F)$ *is p-convex.*

PROOF. Let $\lambda\in\mathrm{Der}(K/F)$. Take any element x of K. If x is in F, then $\lambda(x)=0=\mu(x)$ for any $\mu\in V$. Suppose that x is not in F. Let $E_x=K^p(x)$. Then V restricted to E_x must be a nonzero vector subspace of $D(E_x, K)$, the set of all derivations of E_x into K. Denote by

$V(E_x, K)$ the restriction of V to E_x. Since $[E_x: K^p]=p$, $D(E_x, K)$ is of dimension one over K [1, p. 182], so that $V(E_x, K)=D(E_x, K)$. This shows that $\lambda=\mu$ on E_x for some $\mu\in V$. Therefore, in either case, we have $\lambda=\mu+(\mu-\lambda)\in V+H_x$, which proves the first assertion. Now let F be a subfield of K containing K^p. Let $\lambda\in\bigcap_x(\mathrm{Der}(K/F)+H_x)$. Then for every $x\in K$, there exists an element μ of $\mathrm{Der}(K/F)$ such that $\lambda(x)=\mu(x)$. If $x\in F$, we have $\mu(x)=0$, so that $\lambda\in\mathrm{Der}(K/F)$. This proves the second assertion.

Note that any restricted Lie subring D of $\mathrm{Der}(K)$ of *finite* dimension over K is automatically p-convex. This follows from the second half of Theorem 1 and the fact that $D=\mathrm{Der}(K/\Phi(D))$. However, this is not true in general for *infinite* dimensional restricted Lie subrings.

THEOREM 2. *Suppose that K over F is infinite and purely inseparable of exponent one. Then there exists an infinite dimensional restricted Lie subring D_0 of* $\mathrm{Der}(K)$ *which is not p-convex.*

PROOF. Take a p-basis B of K over F. Of course B is infinite. For every element x_i of B there exists a derivation λ_i in $\mathrm{Der}(K/F)$ such that $\lambda_i(x_i)=1$, while $\lambda_i(x_j)=0$ for any other x_j in B [1, p. 183]. Let D_0 be the vector subspace of $\mathrm{Der}(K)$ spanned by the λ_i over K. Let μ be any derivation in D_0. Then μ vanishes on B, except for a *finite* subset B' of B, and μ^p vanishes on B except for a subset of B'. It follows that $\mu^p\in D_0$. In exactly the same manner one can show that $\lambda\mu-\mu\lambda$ is in D_0 if λ and μ are. Thus D_0 is a restricted Lie subring of $\mathrm{Der}(K)$. Clearly, $\Phi(D_0)=F$, so we must show that $\mathrm{Der}(K/F)$ contains D_0 properly. But this is a trivial consequence of the fact that there exists a μ in $\mathrm{Der}(K/F)$ such that $\mu(x_i)=1$ for every x_i in B [1, p. 181]. Such a μ can not be expressed as a finite linear combination of the x_i. Q.E.D.

REMARK. After finishing this note, the author has been informed that Gerstenhaber [3] has proved that the closedness with respect to the Krull topology, together with the notion of restricted subspace, characterizes the subspace $\mathrm{Der}(K/F)$ of $\mathrm{Der}(K)$.

REFERENCES

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2. M. Gerstenhaber, *On the Galois theory of inseparable extensions*, Bull. Amer. Math. Soc. **70** (1964), 561–566.

3. ———, *On infinite inseparable extensions of exponent one*, Bull. Amer. Math. Soc. **71** (1965), 878–881.

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