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COMPLETELY 0-SIMPLE AND HOMOGENEOUS n REGULAR SEMIGROUPS

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Communicated by E. Hewitt, June 14, 1965

1. In this note we state three new results (Theorems 1, 4 and 5) about the completely 0-simple and homogeneous n regular semigroups.

We follow the notation and terminology of [1] unless stated otherwise. Throughout, S denotes a semigroup with zero. Let $a \in S \backslash 0$. Denote by $V(a)$ the set of all inverses of a in S, that is, $V(a) = (x \in S$: $axa = a$, $xax = x)$. A semigroup S with zero is said to be homogeneous n regular if the cardinal number of the set $V(a)$ of all inverses of a is n for every nonzero element a in S, where n is a fixed positive integer. Let T be a subset of S. We denote by $E(T)$ the set of all idempotents of S in T.

2. The next theorem is a generalization of R. McFadden and Hans Schneider's theorem [3].

THEOREM 1. *Let S be a 0-simple semigroup and let n be a fixed positive integer. Then the following are equivalent.*

(i) *S is a homogeneous n regular and completely 0-simple semigroup.*

(ii) *For every $a \neq 0$ in S there exist precisely n distinct nonzero elements $(x_i)_{i=1}^{n}$ such that $ax_ia = a$ for $i = 1, 2, \cdots, n$ and for all c, d in S $cdc = c \neq 0$ implies $dcd = d$.*

(iii) *For every $a \neq 0$ in S there exist precisely n distinct nonzero idempotents $(e_i)_{i=1}^{h} = E_a$ and k distinct nonzero idempotents $(f_j)_{j=1}^{k} = F_a$ such that $e_ia = a = af_j$ for $i = 1, 2, \cdots, h$, $j = 1, 2, \cdots, k$, $hk = n$, E_a contains every nonzero idempotent which is a left unit of a, F_a contains every nonzero idempotent which is a right unit of a and $E_a \cap F_a$ contains at most one element.*

(iv) *For every $a \neq 0$ in S there exist precisely k nonzero principal right ideals $(R_i)_{i=1}^{k}$ and h nonzero principal left ideals $(L_j)_{j=1}^{h}$ such that*

R_i and L_j contain h and k inverses of a, respectively, every inverse of a is contained in a suitable set $R_i \cap L_j$ for $i=1, \cdots, k$, $j=1, \cdots, h$, and $R_i \cap L_j$ contains at most one nonzero idempotent, where $hk=n$.

(v) *Every nonzero principal right ideal R contains precisely h nonzero idempotents and every nonzero principal left ideal L contains precisely k nonzero idempotents such that $hk=n$, and $R \cap L$ contains at most one nonzero idempotent.*

(vi) *S is completely 0-simple. For every 0-minimal right ideal R there exist precisely h 0-minimal left ideals $(L_i)_{i=1}^{h}$ and for every 0-minimal left ideal L there exist precisely k 0-minimal right ideals $(R_j)_{j=1}^{k}$ such that $LR_j = L_iR = S$, for every $i=1, \cdots, h$, $j=1, \cdots, k$, where $hk=n$.*

(vii) *S is completely 0-simple. Every 0-minimal right ideal R of S is the union of a right group with zero G^0, a union of h disjoint groups except zero, and a zero subsemigroup Z which annihilates the right ideal R on the left and every 0-minimal left ideal L of S is the union of a left group with zero G'^0, a union of k disjoint groups except zero, and a zero subsemigroup Z' which annihilates the left ideal L on the right and $hk=n$.*

(viii) *S contains at least n nonzero distinct idempotents, and for every nonzero idempotent e there exists a set E of n distinct nonzero idempotents of S such that eE is a right zero subsemigroup of S containing precisely h nonzero idempotents, Ee is a left zero subsemigroup of S containing precisely k nonzero idempotents of S, $e(E(S)\backslash E) = (0) = (E(S)\backslash E)e$, and $eE \cap Ee = (e)$, where $hk=n$.*

If $n=1$, then the theorem above takes the same form as R. McFadden and Hans Schneider's theorem [3], except (iv).

3. The following lemmas and Theorem 2 contain main ideas to prove the theorem above.

LEMMA 1. *For all a, b in a Rees matrix semigroup $S = M^0(G; I, \Lambda; P)$, $aba = a \neq 0$ implies $bab = b$. Every completely 0-simple semigroup has this property by Theorem 3.5 [1].*

LEMMA 2. *In a completely 0-simple semigroup S, for a nonzero idempotent e and a nonzero element a in S such that $ea=a$ ($ae=a$) the equation $ax=e$ ($xa=e$) has a solution x in $Se(eS)$. If we denote by x_0 a solution of the equation above, then x_0 is an inverse of a, that is, $ax_0a=a$ and $x_0ax_0=x_0$.*

LEMMA 3. *Let S be a completely 0-simple semigroup. Let $a \in S\backslash 0$. If $E_a = (e_i)_{i=1}^{n}$ and $F_a = (f_j)_{j=1}^{k}$ are sets of all nonzero idempotents of S such that $e_ia = a = af_j$ for every $i=1, 2, \cdots, h$, $j=1, 2, \cdots, k$, $|E_a|$*

$=h$, *and* $|F_a|=k$. *Then* $|V(a)|=hk$.

THEOREM 2. *A nonzero element a in a completely* 0-*simple semigroup S has precisely n inverses if and only if the* 0-*minimal right and left ideals of S containing a contain respectively h and k nonzero idempotents of S such that* $hk=n$.

REMARK. Theorem 2 is a corollary of the following theorem.

THEOREM. *A nonzero element* $a=(g)_{ij}$ *in a Rees matrix semigroup* $S=M^0(G; I, \Lambda; P)$ *has precisely h inverses if and only if* $R_i=((a)_{ij}: a\in G, j\in\Lambda)$ *and* $L_j=((a)_{ij}: a\in G, i\in I)$ *contain precisely h and k nonzero idempotents of S, respectively, with* $hk=n$, *where* $i\in I$, $j\in\Lambda$, $0\neq g\in G$.

Notice that there is no condition of regularity in the theorem.

4. S is said to be $h-k$ type if every nonzero principal left ideal of S contains precisely k nonzero idempotents and every nonzero principal right ideal of S contains precisely h nonzero idempotents of S. A regular semigroup S is said to be $h-k$ regular if S is $h-k$ type. A generalization of P. S. Venkatesan's theorem [5] follows.

THEOREM 3. (1) *A regular semigroup S with zero is* $1-n$ *type in which every nonzero idempotent is primitive if and only if S is the union of its* 0-*minimal ideals each of which is a* $1-n$ *type homogeneous n regular and completely* 0-*simple semigroup.*

(2) *The following statements on a semigroup S with zero are equivalent.*

(i) *S is regular and for any nonzero idempotent e in S the equation* $exe=e$ *has precisely n distinct idempotent solutions* $U(e)=(e_i: i=1, 2, \cdots, n)$ *including e such that e is a right unit of* $U(e)$ *and e is the left zero of* $U(e)$.

(ii) *Every nonzero principal right ideal of S is* 0-*minimal and is generated by just one idempotent. Every nonzero principal left ideal of S is* 0-*minimal and is generated by a nonzero idempotent containing precisely n distinct nonzero idempotents.*

(iii) *For each nonzero a in S there exists a unique set* $U(a)=(a_i: aa_ia=a, i=1, \cdots, n)$ *such that there exist a nonzero principal left ideal containing* $U(a)$ *and n distinct nonzero principal right ideals each of which contains just one element of* $U(a)$. *Every set* $(Sb\cap cS)$ *contains at most one nonzero idempotent, for b, c in S.*

(iv) *For every nonzero element a in S there exist a unique idempotent e and a set* $(f_i: i=1, 2, \cdots, n)$ *of nonzero idempotents such that*

$ea=a=af_i$ $(i=1, 2, \cdots, n)$. Every nonzero principal right ideal contains just one nonzero idempotent and every nonzero principal left ideal contains precisely n nonzero idempotents.

(v) *S is a $1-n$ type regular semigroup and if f is a nonzero idempotent such that $f\notin E(Se\backslash 0)$ then $fE(Se)=E(Se)f=(0)$.*

(vi) *S is a $1-n$ type regular semigroup and for any a, b and c in $S\backslash 0$, $0\neq ab=cb$ implies $a=c$.*

5. W. D. Munn defined the Brandt congruence [4]. A congruence ρ on a semigroup S with zero is called a Brandt congruence if S/ρ is a Brandt semigroup. If S is a $1-n$ (or $n-1$) type homogeneous n regular and completely 0-simple semigroup, then there is a Brandt congruence.

THEOREM 4. *Let S be a $1-n$ type homogeneous n regular and completely 0-simple semigroup. Define a relation ρ on S in such a way that $a\,\rho\,b$ if and only if there exists a set $(e_i)_{i=1}^n$ of n distinct nonzero idempotents such that $e_ia=e_ib\neq 0$, for every $i=1, 2, \cdots, n$. Then ρ is an equivalence on $S\backslash 0$. If we extend ρ on S by defining (0) to be a ρ-class on S, then ρ is a proper Brandt congruence on S. Furthermore, if σ is any proper Brandt congruence on S, then $\rho\subset\sigma$.*

THEOREM 5. *Let S be a $1-n$ regular semigroup in which every nonzero idempotent is primitive. If we define a relation ρ on S by the rule that $a\,\rho\,b$ if and only if there exists a set $(e_i)_{i=1}^n$ of n nonzero idempotents in S such that $e_ia=e_ib\neq 0$ $(i=1, 2, \cdots, n)$. Then ρ is an equivalence on $S\backslash 0$. If we extend ρ to S by defining (0) to be a ρ-class, then ρ is a proper congruence on S such that S/ρ is an inverse semigroup. Furthermore, ρ is the finest such congruence.*

We list more theorems.

THEOREM 6. *If S is a $h-k$ type semigroup with zero and if every nonzero idempotent of S is primiive, then SeS $(e\in E(S\backslash 0))$ is a completely 0-simple and $h-k$ type homogeneous hk regular semigroup.*

THEOREM 7. *Let n, h_n and k_n be positive integers with $h_nk_n=n$. A regular semigroup S with zero is h_n-k_n type in which every nonzero idempotent is primitive if and only if S is a union of its minimal ideals, each of which is a h_n-k_n type homogeneous n regular and completely 0-simple semigroup.*

THEOREM 8. *The following statements on a semigroup S with zero are equivalent (see [2, Theorem 3] and [5]).*

(i) *S is h_n-k_n regular. For all a, x in S $axa=a\neq 0$ implies $xax=x$.*

(ii) *S is $h_n - k_n$ regular. For a, b, x and y in S $xa = sb \neq 0$ and $ay = by \neq 0$ implies $a = b$.*

(iii) *S is $h_n - k_n$ regular. For every e in $E(S\backslash 0)$ there exists a set I of n nonzero idempotents such that eI and Ie are right and left zero subsemigroups of S, respectively, $eI \cap Ie = (e)$ and $e(E(S\backslash I)) = (0) = E(S\backslash I)e$.*

(iv) *Every nonzero principal right (left) ideal of S is 0-minimal and is generated by a nonzero idempotent containing precisely h_n (k_n) nonzero idempotents. $(a \cup aS) \cap (b \cup Sb)$ contains at most one nonzero idempotent, for a, b in S, where $h_n k_n = n$.*

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