

DECISION METHODS IN THE THEORY OF ORDINALS¹

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For an ordinal α , let $RS(\alpha)$, the restricted second order theory of $[\alpha, <]$, be the interpreted formalism containing the first order theory of $[\alpha, <]$ and quantification on monadic predicate variables, ranging over all subsets of α . For a cardinal γ , $RS(\alpha, \gamma)$ is like $RS(\alpha)$, except that the predicate variables are now restricted to range over subsets of α of cardinality less than γ . $\omega = \omega_0$ and ω_1 denote the first two infinite cardinals. In this note I will outline results concerning $RS(\alpha, \omega_0)$, which were obtained in the Spring of 1964 (detailed proofs will appear in [8]), and the corresponding stronger results about $RS(\alpha, \omega_1)$, which were obtained in the Fall of 1964.

The binary expansion of natural numbers can be extended to ordinals. If $x < 2^\alpha$, let ϕx be the finite subset $\{u_1, \dots, u_n\}$ of α , given by $x = 2^{u_1} + \dots + 2^{u_n}$, $u_n < \dots < u_1$. ϕ is a one-to-one map of 2α onto all finite subsets of α . Let Exy stand for $(\exists u)[x = 2^u \wedge u \in \phi y]$, and note that the algorithm $i + j = s$, for addition in binary notation can be expressed in $RS(\alpha, \omega_0)$. It now is easy to see that the first order theory $FT[2^\alpha, +, E]$ is equivalent to $RS(\alpha, \omega_0)$, in the strong sense that the two theories merely differ in the choice of primitive notions; the binary expansion ϕ yields the translation. Similarly, $RS(\alpha, \gamma)$ can be reinterpreted as a first order theory. We will state our results in one of the two forms, and leave it to the reader to translate.

THEOREM 1. *For any α , there is a decision method for truth of sentences in $RS(\alpha, \omega_0)$. The same sentences are true in $RS(\alpha, \omega_0)$ and $RS(\beta, \omega_0)$, if and only if, $\alpha = \beta < \omega^\omega$ or else $\alpha, \beta \geq \omega^\omega$ and have the same ω -tail.*

If $\alpha = z + \omega^y + \omega^n c_n + \dots + \omega^0 c_0$, $y \geq \omega$, then $z + \omega^y$ is called the ω -head of α , and $\omega^n c_n + \dots + \omega^0 c_0$ is called the ω -tail of α .

THEOREM 2. *For any ordinals $\beta > \alpha > \omega^\omega$, $[2^\beta, +, E]$ is an elementary extension of $[2^\alpha, +, E]$, if and only if, α and β have the same ω -tail. The elementary embedding is then given by $h(2^{\alpha_0} x + y) = 2^{\beta_0} x + y$, whereby $x < 2^\tau$, $y < 2^{\alpha_0}$, τ is the common ω -tail of α and β , α_0 and β_0 are respectively the ω -heads of α and β .*

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Let $\alpha = \alpha_0 + \tau \cong \omega^\omega$, where α_0 is the ω -head and τ is the ω -tail of α . From Theorem 2 one easily shows: the ordinals definable in $FT[2^\alpha, +, E]$ (in $FT[2^\alpha, +]$) are those of form $2^{\alpha_0}x + y$, whereby $x < 2^\tau$ and $y < 2^{(\omega^\omega)}$. Actually, Theorems 1 and 2 are but samples of corollaries to Theorem 3, which completely describes the relations on ordinals definable in $FT[2^\alpha, +, E]$.

The results on definability of individuals in $FT[\omega^\alpha, +]$ have been obtained earlier by A. Ehrenfeucht [6]. His methods are quite different; a lucid presentation of this work occurs in [3]. In [3] and [4] it is stated that Ehrenfeucht also knew a decision method for $FT[\omega^\alpha, +]$. However, it seems that nobody has checked out these ideas. The first published proof of the decidability of $FT[\omega, +, E]$, i.e., of $RS(\omega, \omega)$ occurs in [1], and a similar one in [7]. These are both based on my conjecture that $RS(\omega, \omega)$ is just strong enough to express the behavior of finite automata.

The key to the understanding of $RS(\alpha, \omega_0)$ is a natural extension of deterministic finite-state recursions to the transfinite. Let I (input states) and S (internal states) be finite sets. An automaton \mathfrak{A} on I, S consists of an element $A \in S$ (initial state) a map $H: S \times I \rightarrow S$, a map $U: 2^S \rightarrow S$, and a subset $0 \subseteq S$ (the output). Let $\sup_{t < x} (rt)$ stand for the set of all values which the function r takes on cofinal to x , i.e. $Y \in \sup_{t < x} (rt) \cdot \equiv \cdot (\forall z)_0^z (\exists t)_x^z [rt = Y]$. $[A, H, U]$ determines recursively an operator $s[o, \alpha] = \zeta i[o, \alpha]$ from I^α to $S^{\alpha+1}$, namely,

$$\begin{aligned} so &= A, \\ s(x + 1) &= H[sx, ix], \\ sx &= U \left[\sup_{t < x} (st) \right], \quad x \text{ a limit.} \end{aligned}$$

An input sequence $i[o, \alpha]$ is said to be accepted by \mathfrak{A} , in case $s\alpha \in 0$. Extending the proofs given in [1], one now shows,

THEOREM 3. *Let $R(i_1, \dots, i_n)$ be a relation on finite predicates on α . R is definable in $RS(\alpha, \omega_0)$ if and only if there is an automaton \mathfrak{A} such that R consists of those finite (i_1, \dots, i_n) on α , for which the input signal $i[o, \alpha]$ is accepted by \mathfrak{A} .*

In fact there are effective methods, (1) for the construction of \mathfrak{A} from a defining formula Σ of R (synthesis), and (2) for the construction of Σ from \mathfrak{A} (analysis). Theorems 1 and 2 now follow by investigating the behavior of input-free automata.

Let us now consider $RS(\alpha, \omega_1)$. The decidability of $RS(\omega_0, \omega_1)$, i.e., $RS(\omega_0)$ was proved in [2]. It is not difficult to extend the method used

in [2], replacing ordinary automata recursions by transfinite automata. The result is,

THEOREM 1'. *For any countable ordinal α , $RS(\alpha)$ is decidable. For $\alpha < \beta < \omega_1$, $RS(\alpha)$ and $RS(\beta)$ are equivalent if and only if either $\alpha = \beta < \omega^\omega$ or $\alpha, \beta \geq \omega^\omega$ and have the same ω -tail. Furthermore, $RS(\alpha, \omega_1)$ is decidable for any α .*

As in [2] we actually obtain a complete survey over definability in $RS(\alpha, \omega_1)$. In particular, the analog to Theorem 2 holds.

Define the α -behavior of an automaton \mathfrak{A} to be the set $Bh(\mathfrak{A}, \alpha)$ consisting of all input-signals $i[{}_0\alpha]$ which are accepted by \mathfrak{A} . Thus, the ω -behaviors are the ordinary regular sets of finite automata theory.

THEOREM 4. *To any automaton \mathfrak{A} with input (i, j) one can construct an automaton \mathfrak{C} with input i , such that for any $\alpha \leq \omega_1$ and any input-signal i of length $< \alpha$, $i \in Bh(\mathfrak{C}, \alpha) \cdot \equiv \cdot (\exists j)(i, j) \in Beh(\mathfrak{A}, \alpha)$.*

For $\alpha = \omega$ this is the well-known projection-lemma for behaviors of finite automata. The case $\alpha = \omega + 1$ constitutes a significant improvement of the crucial Lemma 9 of [2], and has recently been obtained by R. McNaughton. His construction is very ingenious, and his \mathfrak{C} 's are by far the most intricate finite automata this writer has seen in action. The extension to $\alpha \leq \omega_1$ is an exercise in handling ordinals. Using this improved form of Lemma 9, the definability result of [2] extends as follows,

THEOREM 3'. *To every RS-formula $\Sigma(i_1, \dots, i_n)$ one can construct an automaton \mathfrak{A} , and to every automaton \mathfrak{A} with 2^n -ary input (i_1, \dots, i_n) one can construct an RS-formula $\Sigma(i_1, \dots, i_n)$, such that for any $\alpha < \omega_1$ the behavior $Bh(\mathfrak{A}, \alpha)$ is the relation defined by Σ in $RS(\alpha, \omega_1)$.*

The following problem remains unsolved: Is $RS(\omega_1)$ decidable?

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