THE ISOPERIMETRIC INEQUALITY FOR MULTIPLY-CONNECTED MINIMAL SURFACES[1]

BY JOHANNES C. C. NITSCHE

Communicated by E. Calabi, April 14, 1965

Many proofs have been given of the isoperimetric inequality for minimal surfaces of the type of the disc, which was discovered by T. Carleman [2] in 1921. The question, however, to find a similar inequality for minimal surfaces of higher topological type seems never to have been attacked in the literature. On the basis of new results [3], [4] such an estimate can be derived for multiply-connected minimal surfaces of planar type; and we want to state it here, and sketch the proof, for the case of a doubly-connected minimal surface, answering in part problems 25 and 26 formulated in [5]:

Let S be a minimal surface of the type of the circular annulus of area A (finite or infinite), bounded by two distinct Jordan curves Γ_1 *and* Γ_2 *of lengths* L_1 *and* L_2*, respectively (finite or infinite). If these curves are rectifiable, then the area of S is finite, and the inequality* $(L_1+L_2)^2-4A>0$ *is satisfied.*

The numerical value of the constant 4 can easily be improved. But the question for the best value of this constant—which undoubtedly is 4π—must be left open.

Consider a minimal surface $S=\{\mathfrak{x}=\mathfrak{x}(u,v);\ (u,v)\in\overline{P}\}$, where $\overline{P}$ is the closure of the ring domain $P=\{u,v;\ 0<r_1^2<u^2+v^2<r_2^2<\infty\}$. The vector $\mathfrak{x}(u,v)\in C^2(P)\cap C^0(\overline{P})$ satisfies in P the regularity condition $|\mathfrak{x}_u\times\mathfrak{x}_v|>0$, the condition of vanishing mean curvature $H=0$, and maps the bounding circles of P onto the curves Γ_1 and Γ_2 in a monotonic manner.

The minimal surface has a conformal representation, i.e. a representation where, in addition to having the above properties, the vector $\mathfrak{x}(u,v)$ satisfies in P the relations $\mathfrak{x}_u^2=\mathfrak{x}_v^2$, $\mathfrak{x}_u\cdot\mathfrak{x}_v=0$, and maps the bounding circles of P topologically onto Γ_1 and Γ_2. We set $w=u+iv=\rho e^{i\theta}$, and we shall use interchangeably the notations $\mathfrak{x}(u,v)$ and $\mathfrak{x}(\rho,\theta)$. Once the surface is given in a conformal representation the regularity condition $\mathfrak{x}_u^2>0$ is of no consequence.

For $r_1<r<r_2$ let $\gamma(r)$ be the circle $\{u,v;\ u^2+v^2=r^2\}$, $\Gamma(r)$ its image on S, and $L(r)$ the length of $\Gamma(r)$. Applying a device due to L. Bieberbach [1] and T. Radó [6] it is seen that $L(r)\leqq\operatorname{Max}(L_1,L_2)$.

[1] The preparation of this paper has been supported in part by Air Force Grant AF-AFOSR 883-65.

For $0<\epsilon<(r_2-r_1)/2$ set $r_1'=r_1+\epsilon$, $r_2'=r_2-\epsilon$, and let P_ϵ be the domain $\{w;\ r_1'<|w|<r_2'\}$. Denote by A_ϵ the area of the part of S corresponding to P_ϵ. Integrating by parts we find

$$A_\epsilon = \frac{1}{2}\int\int_{P_\epsilon}(\mathfrak{x}_u^2+\mathfrak{x}_v^2)\,dudv$$
$$= \frac{r_2'}{2}\int_0^{2\pi}\mathfrak{x}(r_2',\theta)\cdot\mathfrak{x}_\rho(r_2',\theta)\,d\theta - \frac{r_1'}{2}\int_0^{2\pi}\mathfrak{x}(r_1',\theta)\cdot\mathfrak{x}_\rho(r_1',\theta)\,d\theta$$

and, using the relation $\rho|\mathfrak{x}_\rho|=|\mathfrak{x}_\theta|$ and estimating,

$$A_\epsilon \leqq \frac{1}{2}\operatorname{Max}(L_1,L_2)\cdot\left\{\operatorname*{Max}_{0\leqq\theta\leqq 2\pi}|\mathfrak{x}(r_1',\theta)| + \operatorname*{Max}_{0\leqq\theta\leqq 2\pi}|\mathfrak{x}(r_2',\theta)|\right\}.$$

All points of S are contained in the convex hull of the curves Γ_1 and Γ_2. Thus A_ϵ is bounded, and so is A.

By a standard argument it now follows that almost all crosscuts, images on S of segments $\{\rho,\theta;\ r_1\leqq\rho\leqq r_2,\ \theta=\theta_0\}$, are of finite length. Let us slit our surface along such a crosscut. We obtain a surface of the type of the circular disc. Its boundary, which might not be a Jordan curve (but this is immaterial here) is rectifiable. By a theorem of M. Tsuji [7] the vectors $\mathfrak{x}(r_1,\theta)$ and $\mathfrak{x}(r_2,\theta)$ are absolutely continuous, and the relations $\lim_{\rho\to r_j}\mathfrak{x}_\theta(\rho,\theta)=\mathfrak{x}_\theta(r_j,\theta)$ $(j=1,2)$ hold for almost all θ. Letting ϵ tend to zero in the expression for A_ϵ we find

$$A \leqq \frac{1}{2}\int_0^{2\pi}|\mathfrak{x}(r_1,\theta)|\,|\mathfrak{x}_\theta(r_1,\theta)|\,d\theta + \frac{1}{2}\int_0^{2\pi}|\mathfrak{x}(r_2,\theta)|\,|\mathfrak{x}_\theta(r_2,\theta)|\,d\theta$$
$$\leqq \frac{1}{2}L_1\cdot\operatorname*{Max}_{0\leqq\theta\leqq 2\pi}|\mathfrak{x}(r_1,\theta)| + \frac{1}{2}L_2\cdot\operatorname*{Max}_{0\leqq\theta\leqq 2\pi}|\mathfrak{x}(r_2,\theta)|.$$

At this point we need an estimate for $\mathfrak{x}(r_1,\theta)$ and $\mathfrak{x}(r_2,\theta)$. Denote by $d>0$ the distance between the curves Γ_1 and Γ_2, and let $\mathfrak{x}_1$ and $\mathfrak{x}_2$ be points on Γ_1 and Γ_2, respectively, for which $|\mathfrak{x}_2-\mathfrak{x}_1|=d$. Assuming $L_1\leqq L_2$ it is easily seen that Γ_1 and Γ_2 are separated by a slab of width $r\geqq d[2\cos(L_2/2d)-1]$. By the theorem of [4] the width of this slab cannot be larger that 3/2 times the larger of the diameters of Γ_1 and Γ_2. This implies $d<L_2$.

Now choose the coordinate system so that $\mathfrak{x}_2$ becomes its origin. Then

$$\operatorname*{Max}_{0\leqq\theta\leqq 2\pi}|\mathfrak{x}(r_1,\theta)| \leqq \frac{1}{2}L_1+d<\frac{1}{2}L_1+L_2,\quad \operatorname*{Max}_{0\leqq\theta\leqq 2\pi}|\mathfrak{x}(r_2,\theta)| \leqq \frac{1}{2}L_2.$$

The asserted inequality follows.

BIBLIOGRAPHY

1. L. Bieberbach, *Über die konforme Kreisabbildung nahezu kreisförmiger Bereiche*, S.-B. Preuss. Akad. Wiss., Math.-Phys. Kl. (1924), 181–188.

2. T. Carleman, *Zur Theorie der Minimalflächen*, Math. Z. **9** (1921), 154–160.

3. J. C. C. Nitsche, *A necessary criterion for the existence of certain minimal surfaces*, J. Math. Mech. **13** (1964), 659–666.

4. ———, *A supplement to the condition of J. Douglas*, Rend. Circ. Mat. Palermo (2) **13** (1964), 192–198.

5. ———, *On new results in the theory of minimal surfaces*, Bull. Amer. Math. Soc. **71** (1965), 195–270.

6. T. Radó, *On Plateau's problem*, Ann. of Math. **31** (1930), 457–469.

7. M. Tsuji, *On a theorem of F. and M. Riesz*, Proc. Imp. Acad. Tokyo **18** (1942), 172–175.

UNIVERSITY OF MINNESOTA