TRIGONOMETRIC SERIES WITH POSITIVE PARTIAL SUMS

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The following problem was proposed by J. E. Littlewood about 15 years ago: Let $S(x)=\sum_{-\infty}^{\infty} c_n e^{inx}$ be a trigonometric series having the property that all its partial sums are positive. Is such a series necessarily a Fourier series? The purpose of this note is to show that such is not the case. It is well known that such a series must be a Fourier-Stieltjes series, and, as was shown by H. Helson, even the weaker condition

$$(1) \qquad \int |S_n(x)|\,dx < \text{const.}, \qquad \left(S_n(x)=\sum_{-n}^{n} c_j e^{ijx}\right)$$

implies $c_n=o(1)$ (cf. Zygmund [2, p. 286]). It has been shown by Mary Weiss [1] that condition (1) does not imply that $S(x)$ is a Fourier series.

LEMMA 1. *There exists a constant $\alpha>0$ such that for every $\epsilon>0$ there exists a real valued trigonometric polynomial $P_\epsilon(x)$, with vanishing constant coefficient, having the properties*:

(i) $|\hat{P}(j)|<\epsilon$,

(ii) *$P_\epsilon(x)>\alpha$ on a set of measure $>\alpha$*,

(iii) *The absolute values of the partial sums of $P_\epsilon(x)$ are less than* 1/2.

PROOF. There exists a constant C such that $|(1/\sqrt{N})\sum_1^N e^{in\log n}e^{inx}| < C$ (cf. Zygmund [2, p. 199]). Take $N>\epsilon^{-2}$ and $P_\epsilon(x) = \operatorname{Re}((1/2C\sqrt{N})\sum_1^N e^{in\log n}e^{inx})$. Properties (i) and (iii) are obvious. Property (ii) follows from the fact that

$$\|P_\epsilon\|_{L^2}=\frac{1}{2\sqrt{(2)}C}, \qquad \sup|P_\epsilon(x)| \leqq \frac{1}{2}\cdot$$

We shall also need the following lemma:

LEMMA 2. *Let $f_j(x)$ be real valued trigonometric polynomials satisfying*:

(a) $\hat{f}(0)=0$,

(b) *$f_j(x)>\epsilon$ on a set of measure $>\alpha$*,

(c) $|f_j(x)|<1/2$.

Then, if $\lambda_j\to\infty$ fast enough, the product

$$(2) \qquad \prod_1^\infty (1 - f_j(\lambda_j x))$$

converges weakly to a singular measure.

PROOF. Our first condition on the growth of λ_n is:

$$(3) \qquad \lambda_n > 3 \text{ times the degree of } \prod_1^{n-1} (1 - f_j(\lambda_j x))$$

which implies that the constant term of $\prod_1^n (1 - f_j(\lambda_j x))$ is 1 for all n. Since the partial products are positive, this implies that the (formal) product (2) is a Fourier-Stieltjes series of a positive measure μ. All that we have to do now is follow the lines of the proof of Theorem V.7.6, p. 209 in Zygmund [2]: We notice first that the partial products $\prod_1^n (1 - f_j(\lambda_j x))$ are partial sums of $S(d\mu)$ followed by long gaps. As is well known, this implies $\prod_1^n (1 - f_j(\lambda_j x)) \to \phi(x)$ a.e. where $\phi(x)dx$ is the absolutely continuous part of μ; but if λ_n grows fast enough (b) implies that the only limit $\prod_1^n (1 - f_j(\lambda_j x))$ can converge to a.e. is zero.

THE EXAMPLE. We take $S(x) = \prod_1^\infty (1 - P_{\epsilon_j}(\lambda_j x))$.

The P_{ϵ_j} are the polynomials defined in Lemma 1, with

$$(4) \qquad 0 < \epsilon_j < 2^{-j-2} \left\| \prod_1^{j-1} (1 - P_{\epsilon_k}(\lambda_k x)) \right\|_A^{-1}$$

(where $\|g\|_A = \sum |\hat{g}(n)|$) and $\lambda_j \to \infty$ rapidly enough so that

(a) $\lambda_j > 3$ times the degree of $\prod_1^{j-1} (1 - P_{\epsilon_k}(\lambda_k x))$ and

(b) $S(x)$ is the Fourier-Stieltjes series of a singular measure (Lemma 2).

From (a) above it follows that a partial sum of $S(x)$ has necessarily the form $\prod_1^q (1 - P_{\epsilon_j}(\lambda_j x))$ times a partial sum of $(1 - P_{\epsilon_{q+1}}(\lambda_{q+1} x))$ plus two groups of terms each having the form

$$P_{\epsilon_{q+1}}(k)e^{ikx} \text{ times some terms from } \prod_1^q (1 - P_{\epsilon_j}(\lambda_j x)).$$

By (iii) $\prod_1^q (1 - P_{\epsilon_j}(\lambda_j x)) > 2^{-q}$ and the partial sums of $(1 - P_{\epsilon_{q+1}}(\lambda_{q+1} x))$ are $> 1/2$ and by (4) the sum of the remaining terms is bounded by 2^{-q-2}, hence the partial sums of $S(x)$ are positive.

REFERENCES

1. M. Weiss, *On a problem of Littlewood*, J. London Math. Soc. **34** (1959), 217–221.
2. A. Zygmund, *Trigonometric series*, Vol. 1, University Press, Cambridge, 1959.

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